Applied Physics Series

THE MECHANICAL PROPERTIES OF FLUIDS

BLACKIE & SON LIMITED
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THE MECHANICAL PROPERTIES OF FLUIDS

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SECOND EDITION

BLACKIE & SON LIMITED
LONDON AND GLASGOW

First issue, 1923
Reprinted 1925
Second edition, revised and enlarged, 1936
Reprinted, with corrections, 1937
Reprinted 1944, 1946

Printed in Great Britain by
Blackie & Son, Limited, Glasgow

PUBLISHERS' NOTE

In recent years a great many researches have been made ıto the mechanical properties of fluids by physicists and ıgineers. These researches are of the utmost practıcal ıportance to engıneers and others, but it is not unusual find that the people who are called upon to apply the sults ın industry have considerable difficulty in findıng ›nnected accounts of the work It is hoped that this llection of essays, many of which are wrıtten by men ıo are the actual pioneers, wıll prove of use in making e recent discoverıes ın the mechanıcal propertıes of fluids ɔre generally known The mathematical notatıon has en made uniform, and the dıfferent chapters have been llated as far as possıble.

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INTRODUCTION

BY ENGINEER VICE-ADMIRAL

SIR GEORGE GOODWIN, K.C.B., LL.D.

(Late Engineer-in-Chief of the Fleet)

To those engaged in the practice of engineering, and ble and willing to utilize the information that mathematical nd physical science can offer them, it is of great assistance) have such information readily available in direct and levant connection with the problems with which they e confronted.

The collective work of this book, issuing as it does om authors highly qualified and esteemed in their respecve fields, whose views and statements will be accepted as ithoritative, supplies concisely and consistently valuable formation respecting the mechanical properties of fluids, id elucidates the evolution of many successful practical)plications from first considerations.

Much has been done in this direction in regard to solids, id this has been assimilated and usefully applied by any; but much less has been produced on the subject of iids, especially in compact form, and this collective work ill doubtless on this account be very welcome.

The necessities of the war brought us face to face with any new problems, a large number of which required it only prompt application of the knowledge available,

but intensive research and rapid development in order to comply with the constantly increasing standards of quality that were demanded. Most of the results are well known: the principles by which they were reached, especially in regard to fluids, are perhaps only vaguely understood, except by a few. The results are certainly appreciated, but further application is probably hindered in many directions for want of this knowledge.

The several contributions to this work enunciate clearly the principles involved, and indicate that a wide field is open for the application of these principles to those who are engaged in industrial avocations and pursuits, as well as to others whose duties continue to be confined exclusively to preparation for war.

Chapters dealing mainly with theoretical considerations form a prelude, clarifying ideas of the physical properties of fluids and providing a sketch of the mathematical theory of fluid motion, with indicators to practical utility.

The sections devoted to practical applications are developed from the underlying theoretical and mathematical considerations. The retention of this method of treatment of the several subjects throughout the work is a valuable feature. These sections will interest a variety of readers, some parts being of particular value to specialists such as the gunnery expert, the naval architect, and the aeronautical engineer, but by far the greater portion of the book will be of general interest to the large body of engineers who have to deal with or use fluids for many purposes in their everyday work.

The chapter on viscosity and lubrication should point the way to the better appreciation and further application of the correct principles of lubrication. The best known present application, that of enabling propeller thrust loads of high intensity to be taken on a single collar, has been highly successful, and it is gratifying to observe that other develop-

nents are already in contemplation, and that some are well dvanced.

The description of the determination of stress by means f soap films is fascinating and deeply interesting; and it is heering to know that certain forms of stress in members f irregular form under load, not amenable to calculation, nd hitherto not determinable and therefore provided for by factor of safety, can now be closely approximated to by xperimental means, and it may be hoped that this or some ther experimental process can be extended practically in the ear future to determine other forms of stress. Success in iis direction would be directly attended with economy of aterial and would facilitate design.

The chapter on submarine signalling indicates the arch of progress in a new branch of engineering, and e author makes the important and significant remark that e science of acoustics shows signs of developing into the igineering stage, a statement worthy of the careful con- deration of all thoughtful engineers.

The section dealing with the wave transmission of energy mes opportunely in view of the large number of practical plications of this form of power transmission that are ing developed, and of those that have matured The same ction gives information respecting the principles govern- g the various forms of flow-meters, and should prove eful to engineers associated with high-power installations io are, by reason of the magnitude of the individual in- illations, being forced to use flow measurements in lieu the definite bulk measurements hitherto favoured by iny, and should give a greater confidence in the use and curacy of flow-meters designed on a sound basis.

The preceding cases are merely mentioned as examples; ery chapter contains a great deal of matter of practical ical engineering interest connected with the mechanical operties of fluids. I have selected these examples as some

of those familiar to me in which I have personally felt the want of some preparatory and explanatory information, such as that given in this book; and it is my recollection that such information was more difficult or inconvenient to obtain in regard to fluids than for solids. My own experiences must, I feel, be those of many others.

The whole series of articles has been to me most interesting, and they show clearly that engineering in the present day requires a great deal of help from pure and experimental science, and is adapting itself to the utilization of branches of science with which it has hitherto not been closely associated. Engineering practice to be worthy of the name must keep itself abreast of and well in touch with those sciences and the developments and discoveries connected with them. This is an onerous task and can only be effected collectively; it is too big for one individual; but works such as this will tend to ease the burden, and convert the task into a pleasing duty.

LIST OF SYMBOLS USED

p or P, pressure.
v or V, volume; velocity.
N, modulus of rigidity.
E, Young's modulus.
t, temperature °C. (or °F.), time.
ρ, density.
s, specific gravity.
κ, bulk modulus; eddy conductivity
T, absolute temperature in °C.
τ, absolute temperature in °F.
C_v, specific heat at constant volume.
C_p, specific heat at constant pressure
γ, surface tension.
β, compressibility.
M, molecular weight.
m, mass.
μ, viscosity.
ν, kinematic viscosity $\equiv \frac{\mu}{\rho}$.
u, v, w, component velocities; displacements
N, Avogadro's constant
ω, specific volume of water; cross section; angular velocity.
S, shearing stress.
$\mathfrak{T}$, twist.
T_q, torque
λ, wave-length; film energy.
R, acoustic resistance.
$\sim$, frequency (cycles per second).

THE MECHANICAL PROPERTIES OF FLUIDS

CHAPTER I

Liquids and Gases

Definitions

We propose to discuss in this chapter some of the more important
neral properties of fluids Common knowledge enables us to
ociate with the terms *fluid*, *liquid*, *vapour*, *gas*, certain properties
ich we regard as fundamental, and which serve to differentiate
se forms of existence from the form which we know as *solid* A
id is, etymologically and physically, *that which flows*, and the
uid or the gaseous state is a special case of the fluid form of exist-
ce A liquid, in general, is only slightly compressible and pos-
ses one free bounding surface when contained in an open vessel.
gas, on the other hand, is easily compressible under ordinary
cumstances, and always fills the vessel which contains it * The
st elementary observation forces upon our notice distinctions
h as these just mentioned, but it still remains to be seen whether
se can be made the basis of a satisfactory classification
Indeed, it is doubtful whether we can make a classification which
l conveniently pigeon-hole the different states of matter, for, as
shall see in the sequel, these different states shade over, under
cial circumstances, one into the other, without the slightest
ach of continuity.
Ordinarily the change from solid to liquid—as when ice becomes

* But compare the quotation on p. 2.

water—or from liquid to vapour—as when water boils—is quite sharp, and the properties of any one substance in the three states are clearly marked off. But substances such as pitch or sealing-wax—behaving under some circumstances as solids, under others as liquids—are distinctly troublesome to the enthusiast for classification. Thus a bell or tuning fork, cast from pitch, will emit a note perfectly clear and distinct as that given by a bell of metal. Nevertheless a block of pitch, left to itself, will in time flow like any ordinary liquid Steel balls placed on the top of pitch contained in a vessel slowly sink to the bottom, and corks placed at the bottom of the vessel will in time appear at the upper surface of the pitch. Such anomalies serve to emphasize the difficulties attendant on any attempt at a rigorous classification. Indeed it is sometimes held that the difference between the solid and liquid states is one of degree, and that all solids in some measure show the properties of liquids However this may be, it is enough to note now that the differences between the solid, liquid, and gaseous states are sufficiently pronounced to make it convenient to attempt a classification which shall emphasize these differences We shall therefore discuss certain properties of matter which serve to define *ideal* solids, liquids, and gases. We shall find that no substances in nature conform to our ideal, which will therefore be but a first approximation to the truth, and later we shall find that small corrections, applied to the equations of state which are the expression of our fundamental definitions, will serve to make the equations represent with considerable accuracy the behaviour of actual substances. This process, involved though it may appear, is both historically correct and physically convenient

Thus the reader may remember that in 1662 the Honourable Robert Boyle took a long glass tube " which by a dexterous hand and the help of a lamp was in such a manner crooked at the bottom that the part turned up was almost parallel to the rest of the tube, and the orifice of this shorter leg . . . being hermetically sealed, the length of it was divided into inches . . . Then putting in as much quicksilver as served to fill the arch, . . we took care, by frequently inclining the tube, so that the air might freely pass from one leg into the other, . . . (we took, I say, care) that the air at last included in the shorter cylinder should be of the same laxity with the rest of the air about it. This done, we began to pour quicksilver into the longer leg, . . till the air in the shorter leg was by condensation reduced to take up but half the space it possessed (I say *possessed* not *filled*) before; we cast our eyes upon the longer leg of the glass, . .

and we observed, not without delight and satisfaction, that the quicksilver in that longer part of the tube was 29 in. higher than the other."*

The pressure and volume of a gas at constant temperature are therefore in reciprocal proportion; that is, at constant temperature the equation of state of a gas is given by

$$pv = k.$$

Succeeding experiments emphasized the truth of this result, and it was not until instrumental methods had advanced considerably that small deviations from this law were shown to exist under ordinary conditions It was then proved that an equation of the type

$$\left(p + \frac{a}{v^2}\right)\left(v - b\right) = k$$

nore closely represented the behaviour of even the more permanent ;ases, later work has shown this equation does not represent with ufficient accuracy the results of experiment, and various other quations of state have, from time to time, been proposed To these quations we shall later have occasion to refer

Again, the *physical* convenience of such a method of approach nay be illustrated by results deduced from the principles of rigid ynamics No body in nature is perfectly rigid—that is, is such that line joining *any* two particles of the body remains invariable in ength during the motion of the body—but considerable simplification f the equations of motion results if we make this assumption, and ne results obtained are in many cases of as high an order of accuracy s is required. We can, if necessary, obtain a closer approximation o the truth by considering the actual deformation suffered by the ody—the problem then becoming one in the theory of elasticity. uppose, for example, that it is our object to deduce the acceleration ue to gravity from observation of the period of a compound pendulum It would be possible to attack the problem, taking into account *b initio* such effects as are due to, say, deformation of the pendulum n its swing, yielding of the supports and the like, afterwards ne lecting such effects as experience has shown to be very small. But ich a method would render the problem almost unbearably complex, esides tending to distract attention from the essentials, and it is oth more convenient and more philosophic to focus one's mind on ne more important issues, to solve the problem first for an ideal

* Boyle's works (Birch's edition, Vol. I, p. 156, 1743).

rigid body, and afterwards to introduce as small corrections effects due to elasticity, viscosity, and so forth.

This, then, is the course which we shall follow in discussing the properties of fluids, and we shall seek, using elastic properties as a guide, definitions which will emphasize those differences which undoubtedly exist between the solid, liquid, and gaseous states of existence.

If we wish to describe completely the elastic behaviour of a crystalline substance, we find that in the most general case twenty-one coefficients are required. For isotropic substances, fortunately, the problem is much simpler, and the coefficients reduce to two, the bulk modulus (κ) and the rigidity modulus (N). These coefficients are

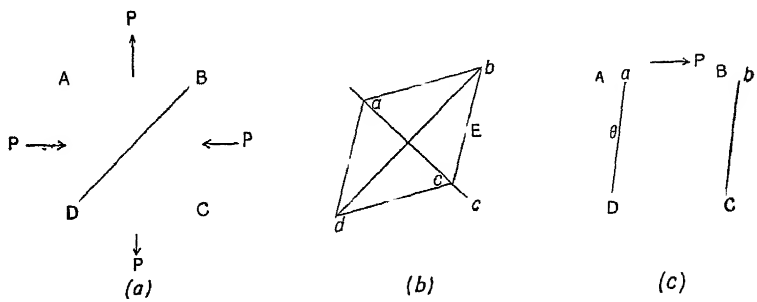


Fig 1

easily specified. Thus, if a cube of unit edge be subjected to a uniform hydrostatic pressure P, so that its volume decreases by an amount δv, the sides of the cube decreasing by an amount e, then, δv being the change in volume per initial unit volume, the ratio of stress to strain, which is the measure of the bulk modulus, is given by $\frac{P}{\delta v} = \kappa$, and, to the first order of small quantities, $\delta v = 3e$.

Suppose now that our unit cube is strained in such a way that in one direction the sides are elongated by an amount e, in a perpendicular direction are contracted by an amount e (fig 1 a), the sides perpendicular to the plane of the drawing being unaltered in length Such a strain is called a shearing strain, and may be supposed to be produced by stresses (P) acting as shown. Considering the rectangular prism BCD, which is in equilibrium under the stresses acting normally over the faces BC and CD, and the forces due to the action on BCD of the portion ABD of the cube, we see that the resultant of the two forces P is a force $P\sqrt{2}$ acting along BD. The

rce due to the action of ABD on BCD must be equal and opposite this. But the area of the face BD is $\sqrt{2}$ units, and there is therere a tangential *stress* of P units, in the sense DB, acting over the agonal area of the cube, due to the action on the prism BCD of e matter in the prism ABD, and called into being by the elastic splacements Thus *the shearing stress, which produces the shearing rain, may be measured by the stress on the areas of purely normal or purely tangential stress.*

If we suppose the directions of the principal axes of shear to be ong the diagonals AC and BD, so that these diagonals are contracted d elongated respectively by an amount e (per unit length), then it n easily be shown that, assuming the strains to be small, the side of e square, the area of the square, and the perpendicular distance tween its sides are, to the first order of small quantities, unaltered the strain. Hence (fig. 1 *b*) the square ABCD strains into the ombus *abcd*, and by rotating the rhombus through the angle *c*EC—ich rotation does not involve the introduction of any *elastic* forces—arrive at the state shown in fig. 1 *c*. Hence, rotation neglected, shearing strain may be regarded as being due to the sliding of allel planes of the solid through horizontal distances which are portional to their vertical distances from a fixed plane DC, the ling being brought about by a tangential stress P applied to the ne AB. The angle θ is taken as a measure of the strain, and the idity modulus (N) is given by the equation

$$N = \frac{P}{\theta}.$$

It is to be remembered that an elastic modulus such as Young's dulus (E) is not independent of κ and N, but is connected with m, as can readily be proved, by the relation

$$E = \frac{9\kappa N}{3\kappa + N}.*$$

We are now in a position to define formally the terms "solid" "fluid".

A solid possesses both rigidity and bulk moduli. If subjected to aring stress or to hydrostatic pressure it takes up a new position quilibrium such that the forces called into existence by the elastic lacements form, with the external applied forces, a system in ilibrium.

* ait, *Properties of Matter*, p 155, or Morley, *Strength of Materials* (1921), p. 11.

A fluid possesses bulk elasticity, but no rigidity. It follows, therefore, that a fluid *cannot permanently resist a tangential stress*, and that, however small the stress may be, the fluid will, in time, sensibly yield to it. In a solid, the stress on an element-plane may have any direction with reference to that plane. It may be purely normal, as on the plane BC (fig. 1 *a*), or purely tangential, as on the plane BD. In a fluid *at rest* the stress on an element-plane must be normal to that plane And it follows at once from this normality, as is proved in all elementary treatises on hydrostatics, that the pressure (p) at a point in a fluid at rest under the action of any forces is independent of the orientation of the element-plane at that point. Thus if x, y, z are the co-ordinates of the point in question,

$$p = \phi(x, y, z).$$

In a *perfect fluid*, no tangential stresses exist, whether the fluid be at rest, or whether its different parts be in motion relative to each other. In all fluids known in nature, tangential stresses tending to damp out this relative motion do exist, persisting as long as the relative motion persists The fluid may be looked on as yielding to these stresses, different fluids yielding at very different time-rates, the *rate* of yield depends on the property known as viscosity

A *perfect liquid* may be defined as an incompressible perfect fluid. No fluid in nature is completely incompressible, and the quantitative study of the bulk moduli of liquids and their relation to other constants of the liquid substance is a matter of great theoretical and practical importance

It must be remembered that the magnitude of the bulk modulus depends on the conditions under which the compression is carried out Two moduli are of primary importance—that in which the temperature of the substance remains constant, and that in which the compression is adiabatic, so that heat neither enters nor leaves the substance under compression Remembering this, we may define a *perfect gas* as a substance whose bulk modulus of isothermal elasticity is numerically equal to its pressure. From this we have at once by definition

$$\frac{dp}{-\dfrac{dv}{v}} = p,$$

or $$pdv + vdp = 0.$$

/Vhence, integrating,

$$pv = k,$$

nd our perfect gas follows Boyle's Law.

Density

Having obtained working definitions of the substances with which 'e have to deal, we proceed to discuss in order certain of their more undamental properties and constants One of the most important f these constants is the *density* of the fluid, defined as the *mass* of nit volume of the fluid. The density of a liquid is accepted, in a hemical laboratory, as one of the tests for its identification, and the nportance in industry of the "gravity" test needs no emphasis. /e shall therefore detail one or two methods for the measurement the density of a liquid—methods for the measurement of gaseous vapour densities are perhaps more appropriately discussed in a eatise on heat.

To determine the density of a substance we have to measure ther (*a*) the mass of a known volume or (*b*) the volume of a known ass Fluids must be weighed in some sort of containing vessel, d if we know the volume of the containing vessel, the measurement the mass of fluid which fills it at a given temperature at once ves us the density of the fluid. The most convenient way of librating a containing vessel is by finding the weight of some uid of known density which fills the vessel at a known temperature nis assumes, of course, that the density of the standard liquid— ually water or mercury—has been determined by some indepndent method, and much laborious research has been done on the asurement of the variation of density with temperature of these o fluids.

Thus, Hälstrom * measured carefully the linear expansion of lass rod, the relation between length and temperature being exssed by the formula

$$L = L_0(1 + at + bt^2).$$

A piece of this rod of volume V was taken and weighed in water different temperatures, the *loss* in weight in water at t° being en by

$$W = W_0(1 + lt + mt^2 + nt^3).$$

* *Ann. Chim. Phys*, 28, p. 56.

The quantities a, b, l, m, n are determined by experiment, and it is clear that the volume at $t°$ of the portion used is

$$V = V_0 (1 + at + bt^2)^3$$

Now since the loss of weight in water is, by Archimedes' principle, equal to the weight of water displaced, and since the volume of this displaced water is equal to the volume of the glass sinker, we have for the density of water at $t°$

$$\rho = \frac{W}{V} = \frac{W_0}{V_0} \cdot \frac{(1 + lt + mt^2 + nt^3)}{(1 + at + bt^2)^3},$$

or

$$\rho = \rho_0(1 + \alpha t + \beta t^2 + \gamma t^3),$$

where α, β, γ are known in terms of l, m, n, a, b. The figures obtained in an actual experiment are quoted below:

$$\left.\begin{aligned} a &= 0{\cdot}000001690 \\ b &= 0{\cdot}000000105 \\ l &= 0{\cdot}0000588 15 \\ m &= -\,0{\cdot}0000062168 \\ n &= 0{\cdot}00000001443 \end{aligned}\right\}\text{whence}\left\{\begin{aligned} \alpha &= 0{\cdot}000052939 \\ \beta &= -\,0\ 0000065322 \\ \gamma &= 0\ 00000001445 \end{aligned}\right.$$

It may be noted in passing that the temperature of maximum density of water may be determined from these results with considerable accuracy. For when ρ is stationary we have $\frac{d\rho}{dt} = 0$, and hence

$$3\gamma t^2 + 2\beta t + \alpha = 0$$

This equation is a quadratic in t, one of the roots is outside the range of the experimental figures, the other is $4{\cdot}108°$ C

From experiments of which this may be quoted as a type, Table I (p. 9) has been drawn up

It will be seen that, if water be used as the calibrating liquid, the determination of the density of a liquid becomes identical with the operation of determining its specific gravity—that is, we find by experiment the ratio of the weight of a certain volume of liquid to that of an equal volume of water *at the same temperature*. The magnitude of this ratio is conveniently denoted by the symbol s_t^t, and may be reduced to density—mass per cubic centimetre—by means of Table I on p. 9. It is more usual to compare the

TABLE I

DENSITY OF WATER IN GM./C.C. AT VARIOUS CENTIGRADE TEMPERATURES

Temperature.	Density.	Temperature.	Density.
Degrees		Degrees.	
0	0·99987	42	0·99147
2	0 99997	44	0 99066
4	1 00000	46	0 98982
6	0 99997	48	0·98896
8	0 99988	50	0 98807
10	0 99973	52	0·98715
12	0 99953	54	0 98621
14	0 99927	56	0 98525
16	0 99897	58	0·98425
18	0·99862	60	0·98324
20	0 99823	62	0 98220
22	0 99780	64	0 98113
24	0 99732	66	0 98005
26	0 99681	68	0 97894
28	0 99626	70	0 97781
30	0 99567	75	0 97489
32	0 99505	80	0 97183
34	0·99440	85	0 96865
36	0 99371	90	0 96534
38	0·99299	95	0 96192
40	0 99224	100	0 95838

eight of the liquid with that of an equal volume of water at 4° C., id this value, s_4^t, may also be deduced from the experimental figures y means of the table. It should be noted that the specific gravities and s_4^t are often doubtful in meaning, for they refer sometimes the ratio of true weights, sometimes to the ratio of apparent eights, no correction being made for displaced air. In experimental ork of high accuracy, it is well both to make this correction, *and indicate that it has been made.*

For ordinary work, the common specific-gravity bottle may be sed, but for precision measurements some form of pyknometer is ecessary The pyknometer is usually a U-tube of small cubic content, s ends terminating in capillary tubes. Three forms are outlined in g. 2. (i) is the original Sprengel type. (ii), a modification introduced y Perkin, possesses several advantages. The instrument, filled by iction, is placed in an inclined position in a thermostat, and excess

liquid is withdrawn from *a* by means of filter paper, until the level in the other limb falls to *b*. The tube is now removed and restored to the vertical position, when the liquid recedes from *a*. If now expansion takes place before weighing, the bulb above *b* acts as a safety space, and all danger of loss by overflow is obviated. The form shown in (iii) was introduced by Stanford, and reduces to a minimum those parts of the vessel not contained in the thermostat, whilst its shape does away with the necessity for suspending wires, as the bottle can be weighed standing upon the balance pan.

In technological practice much specific-gravity work is carried out by means of variable-immersion hydrometers Hydrometer practice and methods can hardly be said to be in a satisfactory state.

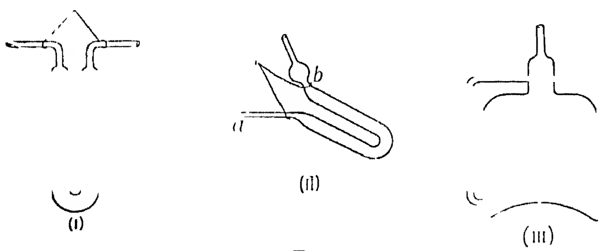


Fig ·

Not only has one to plough through a jungle of arbitrary scales, but the reduction of these scale readings to specific gravities, defined *accurately* as we have defined them already, is no easy matter All hydrometers should carry, marked permanently on their surfaces, some indication of the principle of their graduation, so that their readings may be reduced to s^t_t, s^t_4, or some other definitely known standard For rough work, of course, the arbitrary graduations suffice, and a workman soon learns to associate a reading of, say, x degrees Twaddell with some definite property of the liquid with which he is working. But with more delicate hydrometers an absence of *exact* reference to some definite standard is distinctly unsatisfactory.

Thus the common hydrometer is graduated so that a reading of 1035 corresponds to a specific gravity of 1·035—the standard of reference being very often doubtful—and the Twaddell hydrometer is so constructed that the specific gravity s is given by

$$s = 1{\cdot}000 + 0{\cdot}005\,\tau,$$

where τ is the reading in Twaddell degrees. Clearly on the common

ydrometer the "water-point" is 1000, on the Twaddell hydrometer ero, and, unless the hydrometer carries some reference to the temerature at which its water-point is determined, it becomes impossible atisfactorily to compare the performances of two different hydroneters.

Confusion is worse confounded when we introduce Baumé eadings. In the original Baumé hydrometer water gave the zero oint, and a 15 per cent solution of sodium chloride gave the 15° nark. This for liquids heavier than water. For liquids lighter han water a 15 per cent solution of salt marked the zero point, the water-point being at 10°. Now it is usual to mark the point to which the hydrometer sinks in sulphuric acid of density 1·842 as 66° B We can easily work out a formula of reduction giving the specific gravity in terms of these fixed points. Thus, let

Fig 3

> V be the total volume of hydrometer up to 0°,
> ρ_1, the density of water,
> v, the volume of hydrometer between 0° and n°,
> ρ, the density of liquid in which hydrometer floats at mark n, and
> a, the cross-sectional area of neck

By Archimedes' principle the mass of the hydrometer is given by the two expressions

$$V\rho_1 \text{ and } (V - v)\rho$$

Hence

$$V\rho_1 = (V - v)\rho = (V - an)\rho;$$

and if s is the specific gravity of the liquid,

$$s = \frac{\rho}{\rho_1}.$$

Hence we have

$$\frac{ns}{s - 1} = \frac{V}{a} = k.$$

If we put $s = 1·842$ when $n = 66$, we find that $k = 144·3$, and therefore for any other liquid giving a reading x,

$$s = \frac{144·3}{144·3 - x}.$$

It is obvious that we do not know where we are unless the densities

used in calibration are sharply defined, and the clouds are not appreciably lightened by the practices of Dutch and American hydrometer makers, who take the constant k as 144 and 145 respectively presumably for the convenience of dealing in integral numbers.

Mohr's balance is exceedingly convenient for use in those technological laboratories in which a large number of determinations of density are made. As fig. 4 shows, it is a balance of special form, one arm being divided into ten equal parts and carrying, suspended from a hook by a silk fibre, a glass thermometer which also serves as a sinker The weights provided are in the form of riders, the two

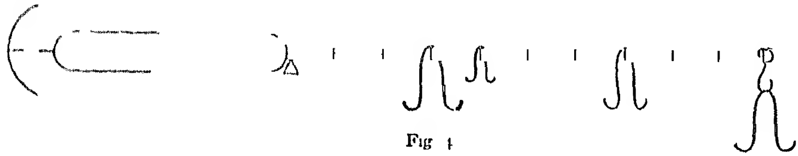


Fig 4

largest being equal, the other two being 0·1 and 0·01 respectively of the largest weights The hook is so adjusted that a body suspended from it is in position at the tenth mark

The other arm of the balance carries a counterpoise with a pointed end, which point, when the balance is in equilibrium, is exactly opposite to a fiducial mark on the fixed support Suppose now that the balance is levelled and is in equilibrium, the sinker being in air Place the sinker in water at 15°. *It will be found that one of the heaviest weights, suspended from the hook, will restore equilibrium* A little thought, based on a knowledge of the law of moments, should convince the student that the specific gravity of a liquid, correct to 0·001, can be read off at once from the positions of the various riders when the balance is in equilibrium, the sinker being immersed in the liquid. Thus the specific gravity of the liquid, the riders being disposed as in fig 4, is 1·374 referred to water at 15°, and may be expressed as a density by means of Table I, p 9.

It happens on occasion that a determination of specific gravity is called for, and that no suitable instruments are at hand. It is worth while knowing that an *accurate* result may be obtained with no more elaborate apparatus than a wooden rod, which need not be uniform, but should be uniformly graduated, and a few counterpoises of unknown weight, made of material unacted on by the liquid under test. Suppose a knitting needle, or a piece of a small triangular

ile, to be fixed in the rod to form a fulcrum (fig. 5) Suspend the veights W and W_1 from the rod by loops of thread, and move W_1

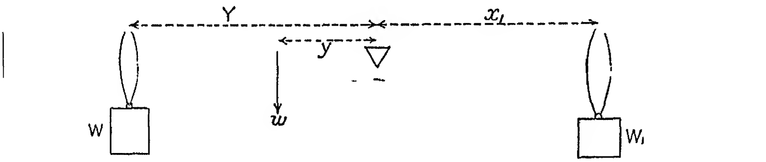


Fig. 5

intil the rod is level. If w be the weight of the rod acting at a listance y from the fulcrum, we have

$$WY + wy = W_1 x_1.$$

f now, without disturbing W, we allow W_1 to hang in a beaker of vater at a temperature t°, we have, if the point of balance be now hifted to x_2, and W_2 be the *apparent* weight of W_1,

$$WY + wy = W_2 x_2$$

These equations give

$$W_1 x_1 = W_2 x_2, \quad \text{or} \quad \frac{W_1}{W_1 - W_2} = \frac{x_2}{x_2 - x_1}.$$

f a beaker of the liquid under test at a temperature t_1° be substituted or the water and the balance point be now at x_3, we have by similar easoning

$$W_1 x_1 = W_3 x_3 \quad \text{or} \quad \frac{W_1}{W_1 - W_3} = \frac{x_3}{x_3 - x_1}.$$

Ience

$$s_t^{t_1} = \frac{W_1 - W_3}{W_1 - W_2} = \frac{x_3 - x_1}{x_3} \cdot \frac{x_2}{x_2 - x_1},$$

nd the specific gravity, which may as before be reduced to density y means of Table I, p. 9, is given accurately in terms of lengths neasured along the rod

The variation of density with temperature has been the subject f many investigations; most of the equations proposed to represent his variation (under certain specified conditions) are applicable, vith great exactness, over limited ranges only. A formula has, however, been put forward recently, which gives the relation between *rthobaric* density and temperature with very considerable accuracy, ver the whole range of existence of the liquid phase. It is developed hus:

We shall see later that, for unassociated liquids, the relation between surface tension and *reduced* temperature (m) is given by

$$\gamma = \gamma_0(1 - m)^n,$$

where n varies slightly from liquid to liquid, but does not deviate greatly from the value 1·2. Further, for any one such liquid the relation between surface tension and the densities of the liquid and vapour phases is

$$\gamma = C(\rho_e - \rho_v)^p,$$

where p does not deviate greatly from the value 4. Eliminating γ between these two equations, we find

$$\rho_e - \rho_v = B(1 - m)^{n/p}$$

where B stands for the p^{th} root of γ_0/C. If we assume that, at the absolute zero, the density of the super-cooled liquid is about four times the critical density, and that of the vapour is negligible, we have

$$\rho_e - \rho_v = 4\rho_c(1 - m)^{0\cdot3},$$

assuming constant values for n and for p. But it is well known that, if we take the mean of the orthobaric densities of liquid and vapour at any temperature, and plot these mean values against temperature, the result is, to a high degree of approximation, a straight line inclined at an obtuse angle to the temperature axis. That is,

$$\rho_e + \rho_v = P - Qm.$$

The condition that, at the critical point ($m = 1$) we have $\rho_e = \rho_v = \rho_c$, combined with the condition previously mentioned that for $m = 0$ we have $\rho_e = 4\rho_c$ and $\rho_v = 0$, gives us

$$\rho_e + \rho_v = 4\rho_c - 2\rho_c m.$$

Taking these equations for $(\rho_e + \rho_v)$ and $(\rho_e - \rho_v)$, eliminating ρ_v and dropping the subscript l, we find

$$\rho = 2\rho_c[(1 - m)^{0\cdot3} + (1 - 0\cdot5m)],$$

which is a reduced equation between density and temperature applicable to all unassociated substances. The equation may be tested by writing it in the form

$$\frac{\rho}{2\rho_c} - (1 - 0\cdot5m) = (1 - m)^s,$$

or

$$Y = X^s,$$

where s may or may not be equal to 0·3 A logarithmic plot of Y against X shows, in general, very good straight lines, whose slope deviates very little from the value 0·3. *But the lines do not pass through the origin.*

It follows then that

$$Y = GX^s$$

$$\rho = 2\rho_c[G(1 - m)^{0\,3} + (1 - 0\,5m)],$$

where G is a constant, whose value varies from liquid to liquid. The variation is not great, and a mean value of G is about 0·91 This general form gives very satisfactory results, but, if very close agreement with the experimental figures is necessary, the values of G and of s special to the particular liquid must be chosen

If we put $m = 0$ in this equation, and take the value of G as 0·91, we see that the (absolute) zero density of the supercooled liquid is about $3\,82\rho_c$, probably a better approximation than the usual value $4\rho_c$

Again, the equation enables us to compute a reasonably good value for the critical density if the density at any one *reduced* temperature is known This, of course, involves a knowledge of the critical temperature If the critical temperature is not known we may make use of Guldberg's rule, that for unassociated liquids the boiling-point under normal pressure is very approximately two-thirds of the critical temperature Putting, therefore, $m = \frac{2}{3}$, we have

$$\rho_b = 2\rho_c[0\,91(1 - \tfrac{2}{3})^{0\,3} + (1 - 0\,5 \times \tfrac{2}{3})],$$

or

$$\rho_b = 2{\cdot}642\rho_c,$$

a general relation between the density at the normal boiling-point and the critical density

By eliminating ρ_c instead of ρ_v a similar formula may be derived to show the march with temperature of the density of the saturated vapour.

Compressibility

We have seen that a perfect liquid is, as a matter of definition, incompressible—that is, its bulk modulus is infinite. Liquids in nature are under ordinary circumstances very slightly compressible,* and the determination of their compressibilities is, in effect, a de-

* Constantinesco, p. 222.

termination of their bulk moduli which, at any given pressure and temperature, is defined by the equation

$$\kappa = \delta p \Big/ \frac{\delta v}{v} = v\left(\frac{\partial p}{\partial v}\right)_T,^*$$

where δp is the additional stress (i e pressure per unit area) causing a *decrease* in volume δv of a substance whose initial volume is v, and $\left(\frac{\partial p}{\partial v}\right)_T$ stands for the rate of decrease of pressure with volume under isothermal conditions (T constant). The compressibility, at any given pressure and temperature, may be defined as the reciprocal of the bulk modulus, i e. the ratio

$$\frac{\delta v}{v}\Big/\delta p = \frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$$

Another definition of compressibility is sometimes used, namely, $\left(\frac{\partial v}{\partial p}\right)_T$; in this case v is the volume of unit mass of the liquid. We shall here confine ourselves to a discussion of the compressibilities of liquids—those of gases and vapours will be treated later. It is clear that a complete study of the compressibility of a liquid resolves itself into the drawing of a p, v, T surface for the substance in question, so that the volume of 1 gm of the substance is known at any pressure and temperature The importance of this knowledge can hardly be overestimated When we have drawn the p, v, T surface for any liquid we are in a position completely to determine its most important thermodynamic properties In this connection the recent work of Bridgman † is pre-eminent in value, and we shall here give a discussion, as brief as may be, of his work, leaving the reader to study details of the older experiments, if he be so minded, in other books The principles involved are simple, but it must be remembered that the experimental difficulties, when the pressures are pushed up to the order of 20,000 Kgm per square centimetre, are very great.

The substance under test is placed in a strong chrome-vanadium steel cylinder, and the pressure is produced by the advance of a piston of known cross-section, the amount of advance of the piston

* This notation means that p is regarded as a function of v and T, and T is kept constant in finding the derivative $\frac{\partial p}{\partial v}$

† *Proc. Amer. Acad. Sci.*, 48, 309 (1912).

giving the change in volume. It would make the story too long were we to discuss in detail the method of packing of the piston to ensure freedom from leakage, and the manner of correction for the change in volume of the cylinder, but it may be of interest to note that the pressure was measured by the change of electrical resistance of a coil of manganin. The resistance of the coil was about 100 ohms, and it was constructed of wire, seasoned under pressure, of resistance 30 ohms per metre. For high-pressure measurements this forms a very simple and convenient form of gauge. It must, of course, be calibrated, and Bridgman performed this by making, once for all, a series of measurements of the change of electrical resistance of the wire with pressure, measuring the pressure by means of a specially constructed absolute gauge It was found that the change of resistance with pressure was so accurately linear up to pressures of 12,000 Kgm. per square centimetre that the readings could be extrapolated with confidence up to 20,000 Kgm. The changes of resistance were measured on a specially constructed Carey Foster bridge

The whole apparatus was immersed in a thermostat, and series of pressure-volume readings were taken at different temperatures From these readings Table II was drawn up, exhibiting the behaviour of water up to 12,500 Kgm per square centimetre pressure, and 80° C.

A careful study of this table will show that we can extract from it data which give very complete details of the thermodynamic properties of water within the range considered. The reader is strongly recommended to work out a few of these results, by so doing he will learn in an hour or two more of the principles of thermodynamics and of the properties of water than he would gain from a week's reading of books where everything is painstakingly explained for him The hints given below should suffice to set him going, and should he have access to Bridgman's papers, it would be well to compare his results with the curves given by Bridgman *

(1) Calculate the compressibility $\left(\frac{\partial v}{\partial p}\right)_T$ or $\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$, and plot a curve between this quantity and p at any one temperature Repeat for various temperatures †

* Loc cit

† The various thermodynamical relations given in (1) to (10) will be found in treatises on thermodynamics, e g *Dictionary of Applied Physics*, article "Thermodynamics". Remember that T here stands for absolute temperature.

(2) Calculate the thermal dilatation $\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p$ or $\frac{1}{v}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p$, and plot it as a function of the pressure at various temperatures

(3) The mechanical work done by the external pressure in compressing the liquid at constant temperature is given by

$$\mathrm{W} = \int p\left(\frac{\partial v}{\partial p}\right)_{\mathrm{T}} dp$$

between given limits, and is obtained by mechanical integration (planimeter, square counting, or the like) of the curves showing the relation between p and v at constant temperature

(4) The total heat given out, Q, during an isothermal compression is similarly derived by mechanical integration from

$$\left(\frac{\partial \mathrm{Q}}{\partial p}\right)_{\mathrm{T}} = -\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p,$$

using the results of (2) to plot the desired curve.

(5) Knowing the mechanical work and the heat liberated in compression, we can find the difference between these, thus giving the change of internal energy along an isothermal, and can plot this against the pressure.

(6) The pressure coefficient is given by

$$\left(\frac{\partial p}{\partial \mathrm{T}}\right)_v = -\frac{\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p}{\left(\frac{\partial v}{\partial p}\right)_{\mathrm{T}}}.$$

It can thus be determined with the aid of the results of (1) and (2), and can be plotted against the pressure

(7) The specific heat at constant pressure may be obtained by mechanical integration from the equation

$$\left(\frac{\partial \mathrm{C}_p}{\partial p}\right)_{\mathrm{T}} = -\mathrm{T}\left(\frac{\partial^2 v}{\partial \mathrm{T}^2}\right)_p$$

This, of course, involves working out the second derivative from the known values of $\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p$ in the same manner as the first derivative is worked out from the original tables. Values of the specific heat as a function of the temperature at atmospheric pressure may be taken, as Bridgman took them, from the steam tables of Marks and Davis.

ATURE

Pressure, Kgm per cm ²	60°	65°	70°	75°.	80°.
0	1 0168	1·0195	1 0224	1 0255	1·0287
500	9965	9992	1·0020	1 0049	1·0075
1,000	9791	·9816	·9842	9869	·9896
1,500	·9632	9657	9682	·9707	9732
2,000	9489	9513	·9537	·9561	·9585
2,500	9363	·9386	9409	9433	9457
3,000	9247	9269	9292	·9314	9337
3,500	9138	9160	9182	·9204	9226
4,000	9037	9058	9080	·9101	9123
4,500	8945	8965	8986	·9008	·9028
5,000	8858	8879	8899	·8920	8940
5,500	8777	8798	8818	8838	8858
6,000	8702	8722	8742	·8762	8781
6,500	·8631	·8650	8670	8689	·8709
7,000	8564	8583	8602	8621	·8640
7,500	·8499	8519	8538	·8557	·8575
8,000	8438	8457	·8477	8495	·8513
8,500	·8381	8400	·8419	8437	·8455
9,000	8327	8346	8364	·8383	8401
9,500	·8275	8294	8313	8331	8349
10,000	·8226	8245	·8264	8282	·8300
10,500	8179	·8198	8216	·8235	8252
11,000	·8133	8152	·8170	·8188	8206
11,500	8088	·8107	·8125	·8143	·8160
12,000	·8043	8062	·8080	8098	·8115
12,500	7999	·8017	·8036	·8054	·8071

(8) Knowing C_p, we can determine C_v from the equation

$$C_p - C_v = -T\frac{\left(\frac{\partial v}{\partial T}\right)_p^2}{\left(\frac{\partial v}{\partial p}\right)_T}.$$

(9) The rise in temperature accompanying an adiabatic change of ressure of 1 Kgm per square centimetre may be deduced with the elp of the equation

$$\left(\frac{\partial T}{\partial p}\right)_\phi = \frac{T}{C_p}\left(\frac{\partial v}{\partial T}\right)_p,$$

l the quantities on the right-hand side of the equation being nown from the results of previous sections. ϕ refers to the ntropy

Finally (10): The difference between the adiabatic and isothermal mpressibilities is given by

$$\left(\frac{\partial v}{\partial p}\right)_\phi - \left(\frac{\partial v}{\partial p}\right)_T = \frac{T}{C_p}\cdot\left(\frac{\partial v}{\partial T}\right)_p^2,$$

d may therefore be calculated

Bridgman has added to the value of this work by making similar idies of twelve organic liquids. For details the student should nsult his original papers *

Interesting relations exist between the compressibility of a liquid d certain other of its physical constants, these we shall discuss er under other heads. Meantime we pass on to a consideration of.

Surface Tension

We may take it as an experimental fact easily deduced from the ost ordinary observations that the surface of a liquid is in a state tension and is the seat of energy. The spherical shape of small ndrops or of small globules of mercury shows that the liquid face tends to become as small as possible in the circumstances, a sphere is that surface which for a given content has the smallest perficial area. Again, the fact that the surface is the seat of energy illustrated by a simple experiment suggested by Clerk Maxwell. agine a large jar containing a mixture of oil and water well shaken

* *Proc. Amer. Acad. Sci.*, **49**, 3 (1913).

up, so that the oil is dispersed through the water in small globules. If the system be left for some time it will be found that the oil has "settled out", and it is clear that the settling-out process has involved the motion of considerable masses of matter—that is, a definite amount of work has been done. The only difference between the two states of the system is that before the settling out the surface-area of the oil-water interface was considerably greater than the area of the interface in the final state. We conclude that the surface possesses *energy*, and it will be seen shortly that an important relation exists between surface energy and surface tension

We assume, then, that across any line of length ds drawn in the surface of a liquid there is exerted a tension γds, the direction of this tension being *normal* to the element ds and in the tangent plane to the surface. The quantity γ is called the *surface tension* of the liquid, its dimensions are clearly those of force ÷ length, and a surface tension is reckoned, in C.G.S. units, in dynes *per centimetre*, or in grammes per second per second. This tension differs from the tension in a sheet of stretched india-rubber, with which it is commonly compared, in that it is, within wide limits, independent of the area of the surface. It is constant at constant temperature, but varies with the temperature, and the calculation of its temperature coefficient is a matter of great theoretical importance. Textbook writers usually give the relation between surface tension and temperature in the form

$$\gamma = \gamma_0(1 - at),$$

a result of little value, holding good over a very limited range The small value attaching to the formula can be shown at once, if we remember that at the critical temperature the surface tension vanishes, so that we must have $t_c = \frac{1}{a}$. But values of t_c calculated in this way are wildly wrong, showing that the range of the formula is exceedingly restricted. It can be shown that, for liquids which do not show molecular complexity, the relation between surface tension and temperature is given by

$$\gamma = \gamma_0(1 - bt)^n,$$

where n varies from liquid to liquid, but in general has a value not differing very greatly from 1·2. This equation holds good from freezing point to critical temperature, and its accuracy may be tested

ɔy comparing the value of t_c obtained from direct experiment with hat obtained from the relation $t_c = \frac{1}{b}$. The test is shown in Table III below.

TABLE III

Substance	n.	b	t_c Calculated Cent Degrees.	t_c Observed Cent Degrees	Difference
Ether . ..	1 248	0 005155	194	193 8	+ 0 2
Benzene .	1·218	0 003472	288	288 5	− 0 5
Carbon tetrachloride	1 206	0·003553	281 5	283 1	− 1 6
Methyl formate .	1·210	0·004695	213	214·0	− 1 0
Propyl formate ..	1 231	0 003774	265	264 9	+ 0 1
Propyl acetate ..	1 294	0 003623	276	276 2	− 0 2

Ve have previously referred to the relation between surface tension ıd surface energy The assumption that the surface tension f a surface is equal to its surface ıergy (per unit area) is another ɔmmon error. The "proof" given sually follows these lines Imagine soap film stretched over the vertical ııe frame shown in fig 6, the bar CD ɜing movable If the bar be pulled ɔwnwards through a distance δx, the oık done against the surface forces $2\gamma l.\delta x$ (remember that the film has ʌo surfaces) But if λ be the energy ıı unit area, since the increase of rface is $2l\delta x$, the increase of superıal energy is $\lambda.2l\delta x$. Hence, equating these quantities, we have

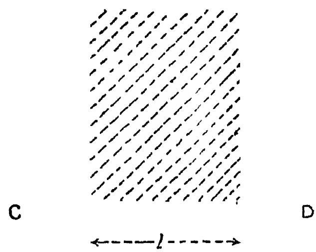


Fig 6

$$\gamma = \lambda.$$

ıt this argument overlooks the important fact that surface tension minishes with increasing temperature. Hence it follows from ermodynamic principles that, in order to stretch the film isotherıılly, heat must flow into the film to keep its temperature constant,

and this heat goes to increase the surface energy. A simple thermodynamic argument shows that the relation between γ and λ is given by

$$\gamma = \lambda + T\frac{\partial\gamma}{\partial T},$$

where T stands for absolute temperature, and only if the temperature coefficient of surface tension were zero would the simpler equation hold.

Since we know the relation between surface tension and temperature for unassociated substances we can easily work out, by substituting for γ and $\frac{\partial\gamma}{\partial T}$ in the equation just given, the relation between surface energy and temperature. This relation is shown in fig 7, the dotted curve showing the variation of surface tension, the full curve the variation of surface energy with temperature. The two curves intersect each other and the axis of temperature at the critical temperature, showing that at that point both surface tension and surface energy vanish. But for lower temperatures the two quantities are in general very different in numerical magnitude, surface energy increasing much faster than surface tension with falling temperature. This important fact should carefully be borne in mind

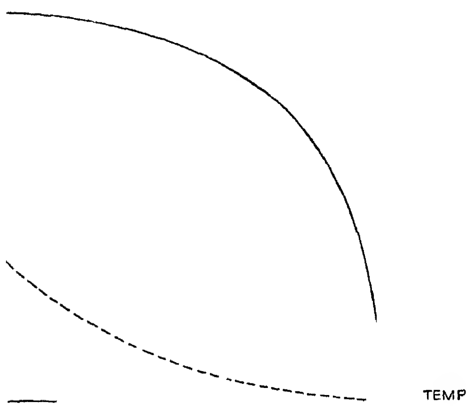


Fig 7

Many relations, empirical and otherwise, have been suggested connecting surface tension with other physical constants Thus, Macleod * has recently found that for any one liquid at different temperatures

$$\gamma = C(\rho_l - \rho_v)^4,$$

* *Trans Faraday Soc*, 1923. The present writer has also shown (*Trans Faraday Soc.*, July, 1923) that the constant C may be expressed in the form $C = \Delta T_c / M^{\frac{2}{3}}\rho_c^{\frac{11}{6}}$, where M is the molecular weight, ρ_c the critical density, and Δ a constant independent of the nature of the liquid

where C is a constant independent of the temperature, ρ_l the density of the liquid, and ρ_v that of the saturated vapour of the liquid.

We should naturally expect surface tension and compressibility (β) to stand in intimate relation, and experiment shows that liquids of high compressibility have low surface tensions and conversely. Richards and Matthews * have examined the quantitative relation between these two constants, and find that, for a large number of unassociated substances, the product $\gamma\beta^{\frac{4}{3}}$ is a constant quantity.

A most important equation connecting surface tension, density, and temperature, is that proposed by Eötvös,†

$$\gamma\left(\frac{M}{\rho}\right)^{\frac{2}{3}} = K(T_c - T - \delta),$$

where M is the molecular weight of the liquid, ρ its density, and δ and K are constants for unassociated liquids, δ being about 6 and K 2 12 The equation shows that a knowledge of the temperature variation of γ enables us to calculate the molecular weight of the liquid under examination, and hence to determine whether its molecules are or are not associated

In recent years this test of association has been slightly altered. Instead of examining the variation of $\gamma\left(\frac{M}{\rho}\right)^{\frac{2}{3}}$ with temperature, the variation of $\lambda\left(\frac{M}{\rho}\right)^{\frac{2}{3}}$ has been studied, where, as we have seen,

$$\lambda = \gamma - T\frac{\partial\gamma}{\partial T}$$

Bennett and Mitchell ‡ have shown that for unassociated liquids this quantity, which we may call the *total* molecular surface energy, is constant over a fairly wide range of temperature, and have used this constancy as a test of non-association

We now turn to the discussion of a problem of fundamental importance—that of the relation between the pressure-excess (positive or negative) on one side of a curved surface and the tension in the surface It is fairly clear that the pressure just inside a curved surface such as that of a spherical bubble is greater than the pressure just outside the surface, and the manner in which pressure-excess is connected with surface tension may be calculated as follows.

* *Zeit Phys Chem*, **61**, 49 (1908)

† See Nernst, *Theoretical Chemistry*, p. 270 (1904).

‡ *Zeit Phys Chem*, **84**, 475 (1913).

Imagine a cylindrical surface whose axis is perpendicular to the plane of the paper, part of the trace of the surface by the plane of the paper being the curve AB (fig. 8). Consider the equilibrium of a portion of this cylindrical surface of unit length perpendicular

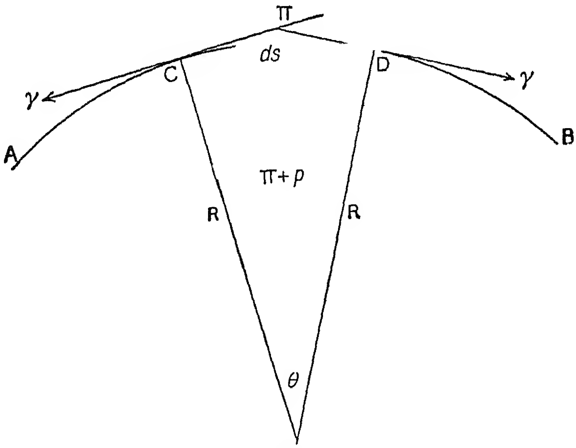


Fig. 8

to the plane of the paper, and of length ds in the plane of the paper. If Π and $\Pi + p$ be the pressures at CD on the two sides of the cylinder, we have, resolving normally,

$$2\gamma \sin\frac{\theta}{2} + \Pi ds = (\Pi + p)ds,$$

when θ is the radian measure of the angle indicated in fig. 8, or, since θ is small,

$$\gamma\theta = pds$$

But $\theta = \dfrac{ds}{R}$ where R is the radius of curvature at C, and therefore

$$p = \frac{\gamma}{R}.$$

If the surface is one of double curvature, the effects are additive and we have

$$p = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where R_1 and R_2 are the principal radii of curvature at the point in question. Thus for a spherical drop, or a spherical air-bubble in a liquid, we have

$$p = \frac{2\gamma}{R},$$

where R is the radius of the drop or bubble. For a spherical *soap-bubble*, which has two surfaces, we should have

$$p = \frac{4\gamma}{R}.$$

The use of the pressure-excess equation, combined with a knowledge of the fact that a liquid meets a solid at a definite angle called the contact-angle, will suffice to solve many important surface-tension problems. Thus the rise or fall of liquids in capillary tubes is readily explained. Water, for example, meets glass at a zero contact

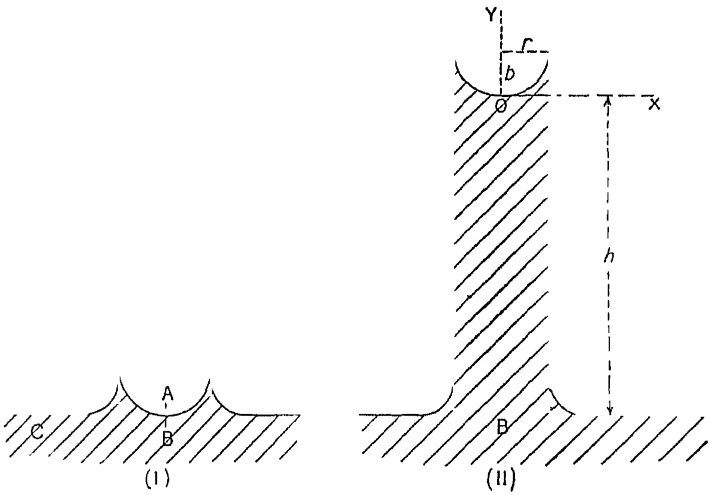


Fig 9

angle, hence the surface of water in a capillary tube must be sharply curved, and the narrower the tube the sharper must be the curvature in order that the liquid may meet the glass at the proper angle The state of affairs shown in fig 9 (1) is impossible, for the pressure at A being atmospheric, the pressure at B must be less than atmospheric by $\frac{2\gamma}{R}$, where R is the radius of curvature of the meniscus at B But the pressure at C in a liquid at rest must be equal to that at B, and the pressure at C is clearly atmospheric. Hence the liquid must rise in the tube until the additional pressure due to the head h just brings the pressure at B up to atmospheric value. We must have therefore

$$\frac{2\gamma}{R} = g\rho h.$$

If the tube be very narrow—and the criterion of narrowness that is $\frac{r}{h}$ shall be small compared with unity—the meniscus will be a segment of a sphere, and the contact angle being zero we may put $R = r$, the radius of the tube, giving the well-known equation

$$\gamma = \tfrac{1}{2}g\rho rh$$

If $\frac{r}{h}$, though small, be not negligible compared with unity the meniscus will be flattened; a very close approximation to the truth may be obtained by treating the meridional curve as the outline of a semi-ellipse. Suppose the semi-axes of the ellipse to be r and b (fig 9 II). If we take the contact angle as zero, there will be an upward pull of $2\pi r\gamma$ on the liquid in the tube all round the line of contact of the liquid with the glass. Equating this to the weight of liquid raised (including the weight of that in the meniscus) we have

$$2\pi r\gamma = \pi r^2 h\rho g + \tfrac{1}{3}\pi r^2 b\rho g$$

or

$$2a^2 = rh + \tfrac{1}{3}rb,$$

if for brevity we write a^2 for $\frac{\gamma}{g\rho}$. But if R be the radius of curvature at O, we have *accurately*

$$\text{Pressure-excess} = g\rho h = \frac{2\gamma}{R},$$

or

$$2a^2 = Rh.$$

Now R, the radius of curvature at the end of the semi-axis minor of an ellipse, is equal to $\frac{r^2}{b}$. Hence

$$b = \frac{r^2 h}{2a^2},$$

and therefore

$$2a^2 = rh + \tfrac{1}{3}r.\frac{r^2 h}{2a^2},$$

or

$$12a^4 - 6rha^2 - r^3 h = 0.$$

Solving this as a quadratic in a^2 and expanding the surd, we obtain

$$2a^2 = rh\left(1 + \tfrac{1}{3}\frac{r}{h} - 0{\cdot}1111\frac{r^2}{h^2} + 0{\cdot}0741\frac{r^3}{h^3}\ldots\right).$$

In all practical cases $\frac{r}{h}$ is small compared with unity, and the above

quation gives values for a^2 (and therefore for γ) in close numerical greement with those obtained from the equation

$$2a^2 = rh\left(1 + \frac{1}{3}\frac{r}{h} - 0{\cdot}1288\frac{r^2}{h^2} + 0{\cdot}1312\frac{r^3}{h^3}\right),$$

btained by the late Lord Rayleigh * as the result of a rather complex nd difficult analysis.

The problem of the measurement of interfacial tensions has ecently assumed great technological importance, mainly on account the rapid development of colloid chemistry and physics. Tanning, yeing, dairy chemistry, the chemistry of paints, oils, and varnishes, gums and of gelatine are all concerned deeply with the properties colloidal systems in which one phase is dispersed in very small articles through the substance of another phase. There is consequently a relatively great extent of surface developed between the vo phases, and the interfacial tension at the surface of separation of le phases may play an important, not to say decisive, part in determining the behaviour of the system

This tension may be measured by a modification of the capillary ibe experiment described above For example, the tension at a enzene-water interface has been measured by surrounding the apillary with a wider tube, and filling with benzene the space reviously occupied by air

Exact determinations can conveniently be made by the dropeight method, wherein a drop of liquid is formed at the end of nd detached *slowly* from a vertical thick-walled capillary tube nmersed in the second (and lighter) liquid The method can, of ourse, be used to determine liquid-air tensions So many erroneous atements have been made concerning the practice and theory of is method that it is worth while considering it in some detail or example, a common practice in physico-chemical works is to quate the weight of the detached drop to $2\pi r\gamma$, a procedure which, ut for the fact that the drop-weight method is often used as a comparative one, would give results about 100 per cent in error. Those, gain, who are alive to the error of writing

$$mg = 2\pi r\gamma$$

ot infrequently tell us that the constant 2π must be replaced by nother constant of value 3 8, for no very apparent or adequate eason. Let us then investigate the problem as exactly as may be

* *Proc. Roy. Soc.* 92 (A), 184 (1915).

in an elementary manner, and see if some justification exists for this procedure. Suppose for the moment that the drop is formed in air. If we assume that the drop is *cylindrical* at the level AB (fig. 10), then, Π being the atmospheric pressure, the pressure at any point in the plane AB is $\Pi + \frac{\gamma}{r}$. Consequently, resolving vertically for the forces acting on the portion of the drop below AB, we have

$$mg + \left(\Pi + \frac{\gamma}{r}\right)\pi r^2 = 2\pi r\gamma + \Pi.\pi r^2,$$

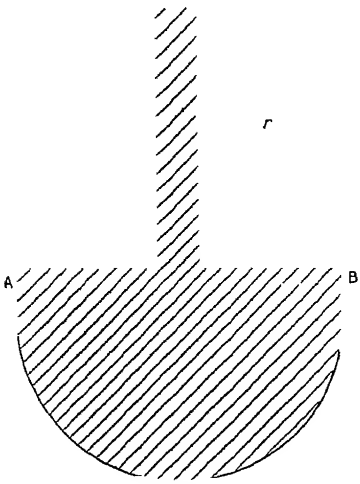


Fig 10

leading to

$$mg = \pi r\gamma,$$

exactly half the value of the weight of a drop as given by most of the textbooks.

But the detachment of a drop is essentially a dynamical phenomenon, and no statical treatment can be complete. We can, however, obtain some assistance from the theory of dimensions. Assume that the mass of a detached drop depends on the surface tension and the density of the liquid, the radius of the tube, and the acceleration due to gravity. We may thus write

$$m = K\gamma^x g^y \rho^z r^w.$$

Dimensionally

$$[M] = [MT^{-2}]^x[LT^{-2}]^y[ML^{-3}]^z[L]^w,$$

leading to

$$x + z = 1, \; x + y = 0, \; y - 3z + w = 0.$$

Solving for w, y, and z in terms of x we find

$$m = K \cdot \frac{\gamma r}{g}\left(\frac{\gamma}{g\rho r^2}\right)^{x-1},$$

or

$$m = \frac{\gamma r}{g} F\left(\frac{\gamma}{g\rho r^2}\right),$$

vhere F is some arbitrary function of the variable $\frac{\gamma}{g\rho r^2}$. The late ord Rayleigh determined the weight of drops of water let fall slowly rom tubes of various external diameters. Knowing the surface ension of water, he was enabled to tabulate the variation of the unction F with that of the independent variable $\frac{\gamma}{g\rho r^2}$; for, as we see, he function F is given by

$$F = \frac{mg}{\gamma r}.$$

n this way the following table was drawn up.

TABLE IV

$\frac{\gamma}{g\rho r^2}$	$F\left(\equiv \frac{mg}{\gamma r}\right)$.
2·58	4·13
1·16	3 97
0 708	3 80
0·441	3 73
0·277	3·78
0 220	3 90
0 169	4 06

will be seen that for a considerable variation of the variable $\frac{\gamma}{g\rho r^2}$ -and this means a considerable variation of r—the function F does not uctuate seriously, and for most purposes it is permissible to assume iat F is constant and equal to 3·8. Hence the reason for the juation

$$mg = 3\ 8r\gamma.$$

The argument for interfacial tensions follows identical lines, and ie reader should have no difficulty in working it out for himself, membering that the drop of density ρ, say, is now supposed to be endent in a lighter liquid of density ρ_1.

If we assume that the liquids with which we are dealing obey the ower law for the variation of surface tension with temperature, we ive

$$\gamma = \gamma_0(1 - m)^n,$$

where, for convenience, we express temperatures in the reduced form. The total surface energy λ is given by

$$\lambda = \gamma - m\frac{\partial \gamma}{\partial m},$$

and, with this form for γ, is

$$\lambda = \gamma_0(1 - m)^{n-1}\{1 + (n - 1)m\},$$

and we see that, contrary to some statements, there is no indication of a maximum value for λ, the march of λ with temperature following the curve shown in fig 7.

We have seen that a reduced equation may be developed between orthobaric density and temperature which, in its simplest form, may be written

$$\rho = 2\rho_c[(1 - m)^{0\cdot 3} + 1 - 0{\cdot}5m].$$

It follows then that free molecular surface energy (e) defined as $\gamma(M/\rho)^{\frac{2}{3}}$ and total molecular surface energy (E) defined as $\lambda(M/\rho)^{\frac{2}{3}}$, have their variation with temperature at once determined on substituting in these expressions the appropriate expressions for γ, λ and ρ

The deduction of the equations showing how e and E vary with the temperature is left to the reader, but it may be noted that e is not a linear function of the temperature, nor is E independent of the temperature, although the variation at fairly low temperatures is very small, and the assumption of constancy over ordinary ranges of temperature need lead to no serious error. Nevertheless it is worthy of note that, considering the whole range $m = 0$ to $m = 1$, the quantity E rises very slowly to a not very pronounced maximum at a temperature about $\frac{4}{5}$ of the critical value, thereafter falling rapidly to zero at the critical point It is interesting to see that this slight maximum is shown in the experimental figures, but was overlooked, as workers in the subject were looking rather for constancy than variation with temperature.

Some time ago Katayama remarked that very considerable simplification resulted if the difference of the liquid and vapour densities were substituted for the liquid density in the definitions of E and e. We thus have

$$e = \gamma\left(\frac{M}{\rho_e - \rho_v}\right)^{\frac{2}{3}} \quad \text{and} \quad E = \lambda\left(\frac{M}{\rho_e - \rho_v}\right)^{\frac{2}{3}},$$

and Katayama points out that, in these circumstances, e and E are linear functions of the temperature given by

$$e = e_0(1 - m), \quad E = E_0(1 + 0{\cdot}2m).$$

As the reader may easily convince himself, these results depend on the power law being followed with n equal to 1·2, and Macleod's law being obeyed with the index equal to 4

If we write this latter law in the more general form

$$\gamma = C(\rho_e - \rho_v)^p,$$

and do not assume any special value for n in the expression for the power law, we readily find that $E = E_0(1 - m)^x\{1 + (n - 1)m\}$, where for brevity x is written for $(n - 1 - 2n/3p)$ If $n = 1{\cdot}2$ and $p = 4$, we have $x = 0$, and Katayama's value for E results In no instances that we have examined is this exactly true The index x is small but positive, and the result is that E climbs by an almost linear ascent to a definite maximum, thereafter falling very rapidly to zero at the critical point—behaviour much more consonant with our usual conception of surface energy than that given by Katayama's equation, which gives E its highest value at the critical point

To establish these results is not difficult If we have, for a substance whose critical temperature is known, a series of values of surface tensions determined over a wide temperature range, a logarithmic plot of $(1 - m)$ and γ serves to test the power law, and to determine the value of n where the power law is followed The values of λ, e and E at different temperatures may then readily be computed E_0, the zero value of the total molecular surface energy, is readily deduced, and it may be remarked that this quantity varies in very interesting and regular fashion with variation in chemical constitution

The quantity $M\gamma^{\frac{1}{4}}/(\rho_e - \rho_v)$ (where M stands for molecular weight) has been named the *parachor.* It provides us with a number which measures the molecular volume of a liquid *at a temperature at which the surface tension is unity*, and therefore gives a most valuable means of comparing molecular volumes under corresponding conditions.

Viscosity

We have seen that a perfect fluid is one in which tangential stresses do not exist, whether the fluid be at rest, or whether its different portions be in motion relative to each other. Such stresses do,

however, appear in all known fluids when relative motion exists, and the fluid may be looked upon as yielding under the stress, different fluids yielding at very different rates.

The most obvious effect of the existence of such tangential stresses between different parts of the fluid is the tendency to damp out relative motion. Thus, if we have a layer of liquid flowing over a plane solid surface, the flow taking place in parallel horizontal layers, the layer of liquid in contact with the surface will be at rest, and there will be a steady increase, with increase of height above the solid surface, in the horizontal velocity of the successive layers. Considering the surface of separation between any two layers, the tangential stress existing there will tend to retard the faster moving upper layer, and to accelerate the slower moving lower layer. The magnitude of the tangential stress may be written down if we assume, following Newton, that the tangential stress is proportional to the velocity *gradient*, so that, if the horizontal velocity is v at a vertical distance y from the fixed surface, we have

$$S \propto \frac{dv}{dy}, \text{ i e. equals } \mu\frac{dv}{dy},$$

where μ is a constant called the coefficient of viscosity of the fluid. If $\frac{dv}{dy}$ is unity, then $S = \mu$. Hence we are led to Maxwell's well-known definition of μ "The viscosity of a substance is measured by the tangential force on unit area of either of two horizontal planes of indefinite extent at unit distance apart, one of which is fixed, while the other moves with unit velocity, the space between being filled with the viscous substance"

The dimensions of μ are those of stress divided by velocity gradient; this works out to

$$[\mu] = [ML^{-1}T^{-1}],$$

so that a coefficient of viscosity in C.G.S. units is correctly given as x gm. per centimetre per second.

If in fig. 1 (c), p 4, we put $Aa = dx$, $AD = dy$, we see that the rigidity modulus (N) is given by

$$S = N\frac{dx}{dy}.$$

Comparing this with $S = \mu\frac{dv}{dy}$, it is clear that the dimensions of

cosity differ from those of rigidity by the time unit, in the same
y as the dimensions of length differ from those of velocity. In
t, the rigidity modulus of a solid determines the *amount* of the
ain set up by a given tangential stress, and the viscosity modulus
a fluid determines the *rate* at which the fluid yields to the stress.

The fall of a sphere through a viscous fluid aptly illustrates
eral interesting physical phenomena; we shall therefore study
problem in some little detail. Considerable assistance is given
an application of the theory of dimensions. Suppose that the
istance (R) experienced by the sphere depends on its radius (a),
velocity (v), and the density (ρ) and viscosity (μ) of the surrounding
id, we then have

$$\mathrm{R} = ka^x \rho^y \mu^z v^w,$$

l as the right-hand side must have the dimensions of a force, we
ain, by equating the exponents of M, L, and T,

$$x = w,\ y = w - 1,\ z = 2 - w,$$

so that

$$\mathrm{R} = ka^w \rho^{w-1} \mu^{2-w} v^w$$
$$= k\left(\frac{va\rho}{\mu}\right)^w \cdot \frac{\mu^2}{\rho}$$

low velocities we may assume that the resistance is proportional
he velocity. Putting therefore $w = 1$, we find

$$\mathrm{R} = k\mu av.$$

nore complex analysis, originally given by Stokes,* shows that

$$\mathrm{R} = 6\pi\mu av$$

however, we assume that for high velocities the resistance varies
he square of the velocity, we have, putting $w = 2$,

$$\mathrm{R} = kv^2a^2\rho,$$

viscosity does not enter into the question—energy is expended,
in overcoming viscous resistance, but in producing turbulent
ion in the liquid.

Returning to the problem of low velocities, let us write down the
ation of motion of a sphere falling vertically through an infinite
n of fluid. The forces acting are—the weight (W) of the sphere

* Lamb, *Hydrodynamics*, p. 532 (1895).

(downwards), the resistance (R), and the buoyancy (B) of the displaced fluid (upwards). This gives

$$W - (B + R) = mf,$$

where m is the mass and f the downward acceleration of the sphere But as the velocity of the sphere increases, R increases *pari passu* so that the acceleration steadily diminishes, until when R has increased to such an extent that

$$W - B = R,$$

f becomes zero, and the sphere henceforward falls with a constant velocity known as the "terminal velocity". Calling this velocity V, the density and radius of the sphere ρ and a respectively, and the density of the fluid ρ_0, we have

$$\tfrac{4}{3}\pi a^3 g(\rho - \rho_0) = 6\pi\mu a V,$$

leading to

$$\mu = \frac{2}{9} \cdot \frac{(\rho - \rho_0)ga^2}{V}.$$

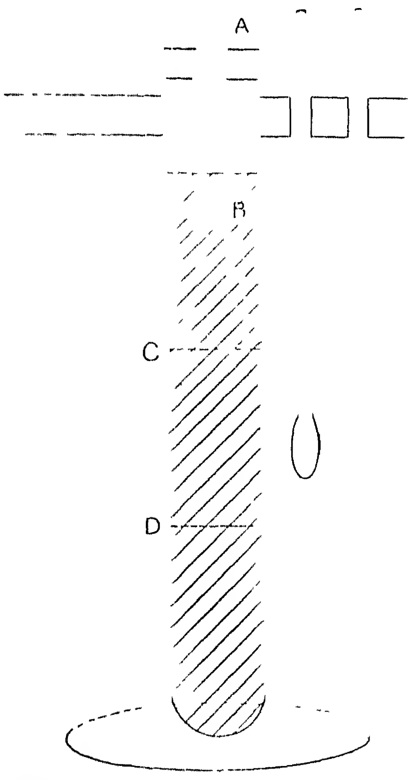


Fig 11

Clearly, measurement of the terminal velocity V enables us to determine the viscosity of a liquid The method is peculiarly suited for the measurement of the viscosities of very viscous liquids such as heavy oils or syrups, and has been much used of late years. The simple apparatus required is shown in fig. 11 The outer cylinder represents a thermostat; the inner cylinder contains the liquid under experiment.

The sphere—steel ball bearings 0·15 cm in diameter are suitable for liquids having viscosities comparable with that of castor oil—is dropped centrally through the tube AB, and its velocity is measured over the surface CD, which represents one-third of the total depth of the liquid.

Two important corrections are necessary—one for " wall-effect ", e for " end-effect "; for it must be remembered that the simple eory given above applies only to slow motion through an *infinite* 'an of fluid.

These corrections have been investigated by Ladenburg,* who s shown that in order to correct for wall-effect we must write

$$V\left(1 + 2{\cdot}4\frac{a}{R}\right) = V_\infty,$$

iere V is the observed velocity, V_∞ the corresponding velocity an infinite medium, and $\frac{a}{R}$ the ratio of the radius of the sphere to it of the cylinder containing the liquid.

Similarly for the end-effect

$$V\left(1 + 3{\cdot}3\frac{a}{h}\right) = V_\infty,$$

iere h represents the total height of the liquid which is supposed be divided into three equal portions, V representing the mean ocity over the middle third. Introducing these corrections into okes's formula, we obtain

$$\mu = \frac{2}{9}\frac{(\rho - \rho_0)ga^2}{V\left(1 + 2{\cdot}4\frac{a}{R}\right)\left(1 + 3{\cdot}3\frac{a}{h}\right)}.$$

The method has been much used during the war period for the 'asurement of μ for liquids of high viscosity, and is fully described a paper by Gibson and Jacobs †

A commercial viscometer has recently come into use, which is of ceedingly simple type, and gives fairly reliable results A steel l $\frac{3}{4}$ in. in diameter is placed inside a hemispherical steel cup of ghtly larger dimensions. The cup carries on its internal surface ee small projections in length about 0·002 in. A little of the oil der examination is poured into the cup, and the ball placed in sition inside the cup. The ball is pressed down on to a table, cup being uppermost, and at a given instant cup and ball are ed clear of the table. The time taken for the ball to detach itself measured, and this gives a measure of the viscosity of the oil.‡

* *Ann der Physik* (IV), **23**, 9 and 447 (1907)
† *Jour. Chem Soc*, **117**, 473 (1920).
‡ For fuller description see Chapter III, p. 119.

Comparative measurements only can be made, and the instrument must be standardized by means of a liquid of known viscosity

Viscometers for use with ordinary liquids usually depend on measurements of the flow of a liquid through a horizontal or vertical capillary tube. The solution of the problem for a horizontal capillary affords an interesting application of the general equations of hydrodynamics, and we shall attack the problem from that side. The reader may or may not be able to follow the arguments by which these equations are established—he may study them at leisure in the treatises of Lamb, of Bassett, or of Webster—what *is* important is that he should see clearly their physical significance and obtain practice in handling them. This is best done by a careful study of one or two of their applications.

The equations of motion of an incompressible fluid are: *

$$\rho\frac{Du}{Dt} = \rho X - \frac{\partial p}{\partial x} + \mu\nabla^2 u, \quad \ldots(\alpha)$$

$$\rho\frac{Dv}{Dt} = \rho Y - \frac{\partial p}{\partial y} + \mu\nabla^2 v, \quad \ldots(\beta)$$

$$\rho\frac{Dw}{Dt} = \rho Z - \frac{\partial p}{\partial z} + \mu\nabla^2 w, \quad \ldots(\gamma)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

p is the pressure at any point, X, Y, Z the components of the external force per unit mass, u, v, w the velocity components.

To apply these equations to the steady flow of a liquid through a horizontal capillary tube, we take the axis of the tube as z-axis, and assume the flow everywhere parallel to this axis Then $u = v = 0$, and from (α) and (β) we have

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0,$$

so that the mean pressure over any section of the tube is uniform.

* See Chapter II, p. 83.

Also from the equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

we have

$$\frac{\partial w}{\partial z} = 0.$$

[ence (α) and (β) vanish, and (γ) becomes, assuming no extraneous ırces,

$$\rho\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0.$$

ince w varies only with t, x, and y, and p only with z, then $\frac{\partial p}{\partial z}$ = constant $= c$ (say), and therefore

$$\rho\frac{\partial w}{\partial t} - \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -c.$$

/e can now obtain the equation known as Poiseuille's equation, for the motion is unaccelerated $\frac{\partial w}{\partial t} = 0$, and

$$\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = c.$$

ʾransforming to polar co-ordinates,*

$$\mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = c,$$

r, w being independent of θ,

$$\mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = c.$$

ʾhis may be written

$$\frac{\mu}{r}\left(r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r}\right) = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = c.$$

ntegrating, we have

$$r\frac{\partial w}{\partial r} = \frac{c}{\mu}\frac{r^2}{2} + A;$$

* See p. 52 *et seq.*

and integrating a second time

$$w = \frac{1}{4} \cdot \frac{c}{\mu} \cdot r^2 + A \log r + B,$$

where A and B are constants of integration When $r = 0$ (on the axis of the tube), w is finite. Consequently A must be zero, and

$$w = \frac{1}{4} \cdot \frac{c}{\mu} \cdot r^2 + B$$

Also, if there is no slipping of the liquid at the walls of the tube, when $r = a$, $w = 0$, and consequently

$$w = -\frac{1}{4} \cdot \frac{c}{\mu} (a^2 - r^2).$$

If V is the *volume* of liquid which escapes from the tube in a time T the volume issuing in unit time is given by

$$\frac{V}{T} = \int_0^a w.2\pi r dr$$

$$= -\frac{2\pi}{4\mu} \cdot c \int_0^a (a^2 - r^2) r dr = -\frac{\pi}{8} \cdot \frac{a^4}{\mu} \cdot c$$

If p_1 and p_2 are the pressures at the entrance to and exit from the pipe, then remembering that $\frac{\partial p}{\partial z}$ stands for the rate of *increase* of p with z, we have

$$c = \frac{\partial p}{\partial z} = -\frac{p_1 - p_2}{l},$$

where l is the length of the pipe. Hence

$$\mu = \frac{\pi}{8} \cdot \frac{a^4 T}{V} \cdot \frac{p_1 - p_2}{l}.$$

If the liquid is supplied to the tube under a constant head h, and escapes into the air at a low velocity, we have

$$\mu = \frac{\pi}{8} \cdot \frac{a^4 T}{V} \frac{g\rho h}{l}.$$

This equation is known as Poiseuille's equation. All the quantities on the right-hand side may be determined experimentally, and hence μ may be evaluated.

Comparative measurements by this method are usually made using twald's viscometer (fig. 12). The bulb C is filled with the liquid der examination, which is then drawn up by suction until it fills : bulb D. The pressure is then released, and the time of transit ween the marks A and B observed. The pressure head is varying oughout the fall, and clearly we cannot apply Poiseuille's equation it stands. But, noting that for liquids of equal densities and ferent viscosities the times will be proportional to the viscosities, 1 that for liquids of equal viscosities and different densities the nes will be inversely proportional to the densities, have in general,

$$t = K\frac{\mu}{\rho} \quad \text{or} \quad \mu = c\rho t,$$

ere c is a constant for the apparatus to be deter- ned by using a liquid of known viscosity.

A viscometer of dimensions suitable for the deter- nation of the viscosity of water is not suited for use h heavy oils But if we have a series A, B, C, . viscometers of gradually increasing bore, calibrate by using water, and then use the most viscous uid suitable for A in order to calibrate B, continuing s process as far as necessary, we are provided with hain of viscometers which can be used over a very le range.

Fig 12

Since the viscosity of liquids decreases rapidly h increase of temperature, it is of vital necessity t the apparatus be enclosed in some form of ther- stat and that the temperature of experiment be en and *recorded.* This rapid change of viscosity with temperature kes it very difficult to obtain relations between the viscosities of ferent chemically related substances, as it is by no means easy to tle the temperature of comparison. It has been found, however, t consistent results may be obtained if viscosities are compared *temperatures of equal slope*—that is at temperatures for which $\frac{d\mu}{dt}$ he same. Using this standard it has been shown, for example, t the molecular viscosities * of a homologous series increase by a istant amount for each addition of CH_2

* The molecular viscosity of a liquid is defined as $\mu(Mv)^{\frac{2}{3}}$, where M is the ecular weight and v the specific volume of the fluid concerned.

There are many important practical problems which depend for their solution on a knowledge of friction in fluids The viscosities of mixtures of liquids, the viscosities of gases, the theory of lubrication the discussion of turbulent motion, to mention but a few, present important and most interesting aspects. These matters are fully discussed in Chapter III

One interesting problem may be mentioned in passing—the suspension of clouds in air, where we have the apparent paradox of a fluid of specific gravity unity suspended in a fluid of specific gravity 0·0013. The paradox is cleared up by an application of Stokes' formula,

$$V = \frac{2}{9}\frac{(\rho - \rho_0)ga^2}{\mu}.$$

Taking the viscosity of air as 0·00017 in C.G S units, the reader is recommended to calculate the terminal velocities of spheres of water say 0·1, 0·01, . . . cm in radius The terminal velocities of minute drops will be found to be surprisingly small

The kinetic theory of liquid viscosity has not received a great deal of serious attention, and formulæ developed to show, for example, the dependence of liquid viscosity on temperature have usually a purely empirical basis Of these, one proposed by Porter may be specially noted. Suppose that the variation of viscosity with temperature has been experimentally investigated for two liquids Take a temperature T_1 at which one liquid has a viscosity η_1 Find the temperature T_2 at which the second liquid has the viscosity η_1 Repeat for different values of T_1. Then T_1/T_2 is a linear function of T_1

Recently Andrade has put forward a kinetic theory in which he assumes " that the viscosity is due to a communication of momentum from layer to layer, as in Maxwell's theory of gaseous viscosity but that this communication of momentum is not effected to any appreciable extent by a movement of the equilibrium position of molecules from one layer to another, but by a temporary union at the periphery of molecules in adjacent layers, due to their large amplitudes of vibration."

This assumption leads to a formula connecting viscosity and temperature of the form

$$\eta = Ae^{\frac{c}{T}},$$

a formula which had been put forward previously on an empirical

ısis. Porter's relatıon, as the reader may verıfy, follows at once om thıs equation.

In the deductıon of thıs equation, varıatıon of volume with mperature has been neglected and, takıng thıs factor into account, ıdrade deduces a second formula,

$$\eta v^{\frac{1}{3}} = \mathrm{A}e^{\frac{c}{v\mathrm{T}}},$$

ıere v ıs the specıfic volume. For a great many organıc lıquıds ıs formula gıves a very good fit, though, as was to be expected, ıter and the tertıary alcohols show abnormalıtıes.

Equations of State

Much labour has been expended on the problem of devısıng uatıons which shall represent accurately the pressure-volume-nperature relatıons of a substance ın ıts lıquıd and ıts gaseous ases It may be saıd at once that ıt ıs ımpossıble to devıse an uatıon whıch shall be accurate over such a range wıthout beıng possıbly cumbrous Nevertheless the sımpler equatıons have, we shall see, consıderable value ın gıvıng a faırly adequate ɔresentatıon of the general behavıour of a homogeneous fluıd.

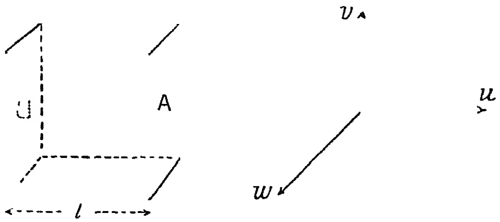


Fig 13

A very sımple type of such a fluıd ıs a gas, consıdered as an emblage of materıal poınts whıch are ın rapıd—and random—ıtıon, and whıch do not exert any attractıve forces on each other nsider a given quantıty of such a gas enclosed ın a cube of sıde ınd let the component velocıtıes of any one partıcle be u, v, w shown (fig. 13).

The pressures on the faces of the cube are due to the ımpacts of partıcles thereon. At any ımpact on, say, the face A the velocity nponent normal to that face wıll be reversed, and the change of mentum of the partıcle consequent on the rebound wıll be

$$mu - (-mu) = 2mu.$$

The time of travel from A to B and back is $\frac{2l}{u}$ sec.; the frequency of the impacts on the face A is $\frac{u}{2l}$; hence for any one particle the change of momentum at A per second will be

$$2mu \times \frac{u}{2l} = \frac{mu^2}{l},$$

and similarly for the other particles

The *force* on the face A due to molecular bombardment is, therefore, $\frac{m}{l}\Sigma u^2$.

If p is the *pressure* on the face A,

$$p = \frac{1}{l^2}\,\frac{m}{l}\Sigma u^2 = \frac{m}{l^3}\Sigma u^2 = \frac{m}{V}\Sigma u^2,$$

where V is the volume of the cube. Similarly, for the pressures on the faces perpendicular to A, we have the expressions

$$\frac{m}{V}\Sigma v^2, \quad \frac{m}{V}\Sigma w^2.$$

But these pressures are equal, and therefore

$$p = \frac{m}{V}\Sigma u^2 = \frac{m}{V}\Sigma v^2 = \frac{m}{V}\Sigma w^2$$

$$\tfrac{1}{3}\frac{m}{V}\Sigma(u^2 + v^2 + w^2) = \tfrac{1}{3}\,\frac{m}{V}\Sigma U^2,$$

where $$U^2 = u^2 + v^2 + w^2$$

Now let us define a mean velocity $\overline{U}$ by the relation

$$\overline{U}^2 = \Sigma\frac{U^2}{N},$$

N being the total number of particles in the cube. We then have

$$pV = \tfrac{1}{3}mN\overline{U}^2,$$

and mN being the mass of the gas, we have, if ρ be its density,

$$\rho = \frac{mN}{V}, \quad \text{and} \quad p = \tfrac{1}{3}\rho\overline{U}^2.$$

Hence Boyle's Law.

If we assume that $\overline{U}^2$ is proportional to the absolute temperature,

e have Charles's law, and can write as the characteristic equation of ır " perfect " gas

$$pV = RT.$$

V stands for the volume of unit mass of the gas, R will be different r different substances. A simple deduction from our fundamental uation shows, however, that the gramme-molecular volume* is the ne for all gases. Consider two different gases for which

$$p_1v_1 = \tfrac{1}{3}m_1N_1\overline{U}_1^2 \text{ and } p_2V_2 = \tfrac{1}{3}m_2N_2\overline{U}_2^2.$$

the pressures and volumes are the same,

$$m_1N_1\overline{U}_1^2 = m_2N_2\overline{U}_2^2.$$

the temperatures are equal, then, assuming that the mean kinetic ergies are the same,

$$m_1\overline{U}_1^2 = m_2\overline{U}_2^2,$$

that $$N_1 = N_2$$

at is, *equal volumes of two gases, under the same conditions of ıperature and pressure, contain the same number of molecules*

This is the formal statement of Avogadro's hypothesis. It ows, therefore, that the weights of these equal volumes are protional to the molecular weights of the gases, and hence that the mme-molecular volumes of all gases, measured under the same ıditions of temperature and pressure, are the same.

The gramme-molecular volume, measured at 0° C and 760 mm. Hg, is 22·38 litres. If then V stands for this volume, the constant vill be the same for all gases. Its value should be calculated by reader.

But no gas behaves in this simple manner, although for moderate ssures and high temperatures the equation is accurate enough for inary computations, as far as the more permanent gases are cerned.

Suppose that we study experimentally the $p - v$ relations of erent fluids, drawing the isothermals for various different temıtures (fig. 14). Starting at a sufficiently low temperature we find the volume steadily diminishes with increase of pressure up to a ain point at which the fluid separates into two phases—liquid and eous. The pressure then remains constant until the gaseous

* I e the volume occupied by M gm of a gas at normal temperature and sure, where M is the molecular weight.

phase has completely disappeared, when further increase of pressure causes but small diminution in volume. If we now repeat the experiment at a higher temperature, we find that the horizontal portion AB of the curve, representing the period of transition from the gaseous to the liquid phase, is shorter, and shortens steadily with increasing temperature until the isothermal for a certain temperature exhibits a point of inflexion with a horizontal tangent, running for a moment parallel to the volume-axis, and then turning upwards again. The temperature for which this isothermal is drawn is called the *critical temperature*, and the point G the critical point * Above this temperature no amount of pressure causes a separation into two distinct phases

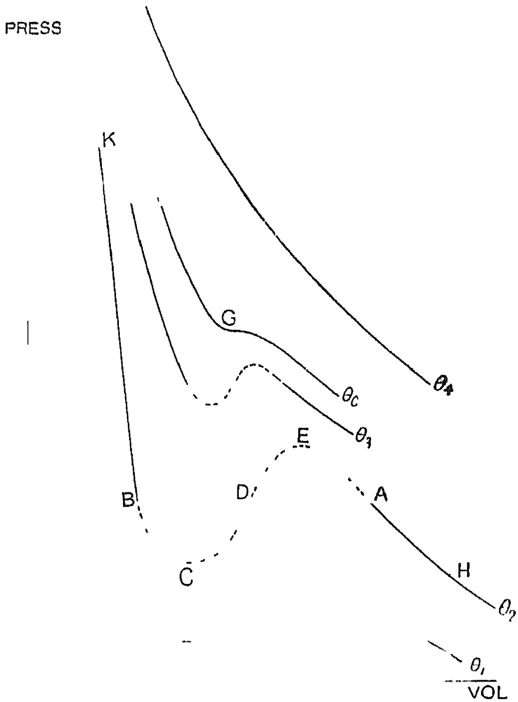


Fig 14

The experimental determination of these curves, over a wide range of pressure and temperature, is a matter of no small difficulty. Once a suitable pressure gauge has been devised—we have seen that the change of electrical resistance of manganin may be utilized—observations are fairly straightforward, but the calibration of such a gauge demands experimental work on a heroic scale Amagat, for example, performed a Boyle's Law experiment in which nitrogen was compressed in the closed (shorter) limb of a U-tube, the open limb being installed on the side of a shaft 327 m. deep. The *p-v* relation for nitrogen being known, this gas may then be used as a standard in studying the behaviour of other

* For a description of the physical state of the fluid at the critical point, consult any of the standard treatises on heat, e g Poynting and Thomson, or Preston.

ases, or in calibiating a different type of pressure gauge.

Let us now examine briefly the character of the curves obtained om experiments of this type. It is convenient to plot pv against , as this procedure exhibits very clearly the departure of the gas oncerned from the "perfect" state. Some of the results obtained e shown in fig. 15

The reason for the difference between the isothermals for an eal and for an imperfect fluid is not far to seek. The equation

$$pV = RT$$

kes no account of the forces of attraction between the molecules, or of the volume occupied by the molecules themselves. It makes

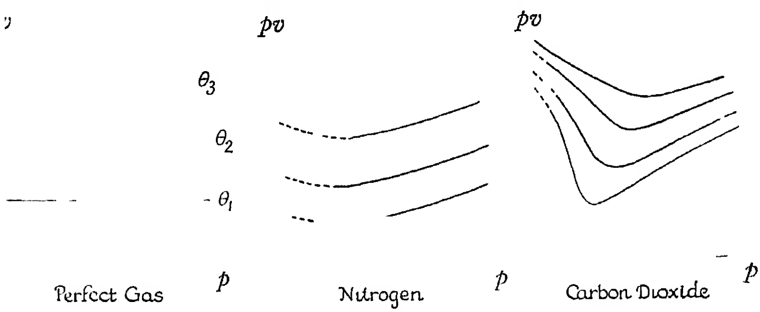


Fig 15

zero when p becomes indefinitely great, and it is clearly more in cordance with the properties of fluids to put

$$p(V - b) = RT,$$

that as p increases indefinitely, V tends to the limit b, b represents, erefore, the smallest volume into which the molecules can be cked.

Further, the mutual attraction of the molecules will result in the oduction of a capillary pressure at the fluid surface, the intensity the molecular bombardment will be diminished, and the pressure the surface of the containing vessel correspondingly decreased. it, therefore,

$$(p + \omega)(V - b) = RT.$$

ithout discussing the matter very closely, we can determine the lue of ω from consideration of the fact that the attraction between o elementary portions of the fluid is jointly proportional to their

masses—that is, in a homogeneous fluid, to the square of the density, or inversely as the square of the volume. We see, then, reasons for writing $\omega = \frac{a}{V^2}$, and the equation of state becomes

$$\left(p + \frac{a}{V^2}\right)\left(V - b\right) = RT,$$

the form originally proposed by van der Waals

This equation is a cubic in v, and if the isothermals are plotted for different values of θ, we obtain curves whose general shape is that of the curve HAEDCBK of fig. 14. It will be observed that for temperatures below the critical temperature a horizontal constant pressure line cuts any given isothermal either in one point or in three points—corresponding to the roots of the van der Waals cubic Taking an isothermal nearer to the critical temperature, we see that the three real roots are more nearly coincident, and at G, the critical point itself, the roots coincide. Above the critical point, a horizontal line cuts any given isothermal in one point only—two of the roots of the cubic are imaginary. If we write down the condition that the three roots shall be coincident, we easily arrive at values of the critical constants in terms of the constants of van der Waals' equation These are

$$v_c = 3b, \quad p_c = \frac{a}{27b^2}, \quad T_c = \frac{8a}{27Rb}.$$

But it is preferable to write down the condition that at the critical point the isothermal has a point of inflexion with a horizontal tangent If we therefore differentiate van der Waals' equation with respect to v, put $\frac{\partial p}{\partial v}$ and $\frac{\partial^2 p}{\partial v^2}$ equal to zero, the resulting equations, combined with the original equation of state, serve to determine p_c, v_c, and T_c The work is left as an exercise to the reader.

This method is preferable, since it is perfectly general and may be applied to characteristic equations which are not cubics in v, and to which, therefore, the "equal-root" method, beloved of writers on physical chemistry, is not applicable

It will be observed that the equation tells us nothing concerning the straight line AB, which represents the actual passage observed in nature from the vapour to the liquid phase. The position of this line on any given isothermal can, however, be obtained from the simple consideration that the areas DBCD and AEDA must be

qual,* and the line must be drawn to fulfil this condition.

Fig 15 shows that the pv-p curves in general exhibit a minimum alue for pv, and that the locus of these points lies on a definite curve. 'he equation to this curve may be obtained by writing pv as y and as x in the characteristic equation, and expressing the condition iat y should have a minimum value.

Of the other characteristic equations that have from time to time een proposed we may cite, naming them by their authors:

Clausius (a):

$$\left(p + \frac{a}{TV^2}\right)(V - b) = RT;$$

Clausius (b).

$$\left(p + \frac{a}{T(V+C)^2}\right)(V - b) = RT;$$

Dieterici (a):

$$\left(p + \frac{a}{V^k}\right)(V - b) = RT,$$

Dieterici (b).

$$p(V - b) = RTe^{-\frac{A}{RTV}}.$$

he deduction of the critical constants from these equations is left the reader.

The value of a characteristic equation which shall closely represent e pressure-volume relations of a fluid over a wide range of pressures id temperatures, is obvious We have seen, in the section on mpressibility, that many important physical constants may be pressed in terms of thermodynamic equations involving certain fferential coefficients and integrals The values of these physical nstants may be worked out by substituting, in the appropriate ermodynamic equations, the values of the differential coefficients

* For in any reversible cycle $(\int)\frac{dQ}{T} = 0$ If the cycle be isothermal,

$$(\int)\frac{dQ}{T} = \frac{1}{T}(\int)\,dQ, \text{ and therefore } (\int)dQ = 0.$$

it for *any* cycle

$$(\int)(dQ + dW) = 0,$$

d hence for an isothermal reversible cycle $(\int)dW = 0$ So that, if we take it mass of the substance reversibly round the cycle AEDCBDA (fig. 14), the rk done, represented by the sum of the positive and negative areas AEDA and CBD, must be zero. Hence the two areas are equal (For critical remarks on is proof see Preston, *Theory of Heat*, 479 (1904), or Jeans, *Dynamical Theory of ises*, p. 159.)

obtained from the differentiation of the equation of state Unfortunately, no equation yet proposed covers the whole ground satisfactorily. An equation which fits the experimental figures at one end of the scale is usually unsatisfactory at the other end, and conversely.

A few simple tests may be suggested by which the fitness of any given equation may be roughly examined. The experiments of Professor Young show that, while the value of the ratio $\frac{RT_c}{p_c v_c}$ varies slightly from substance to substance, its mean value may be taken as about 3·75 Now the equation of state of an ideal gas gives unity for this ratio, and is clearly very far out of it. Van der Waals' equation gives

$$\frac{RT_c}{p_c v_c} = \left(R \times \frac{8a}{27Rb}\right) - \left(\frac{a}{27b^2} \times 3b\right) = 2\cdot67,$$

and is not a very good approximation to the truth

Similarly the (a) and (b) equations of Clausius give for this ratio the values 2·67 and 3·00 respectively, and the corresponding equations of Dieterici give 3·75 and 3·69, if in the (a) equation we take the value of k as $\frac{5}{3}$

Another test may be derived from the experimental fact that the critical specific volume v_c is about four times the liquid volume Now the constant b, which represents the least volume into which the molecules can be packed, cannot be seriously different from the liquid volume. Accordingly we find, on working out the values for v_c, that for the (a) and (b) equations of Clausius the values of v_c are $3b$ and $4b$ respectively,* while the corresponding equations of Dieterici give the values $4b$ and $2b$. For van der Waals' equation the value is, as we have seen, $3b$.

But the subject may be studied from a different point of view Instead of attempting to devise an equation which shall represent the properties of a substance over a wide range—a process which usually results in a cumbrous formula—we may try to arrive at an equation which shall be simple and manageable in form, so that the various physical constants of the fluid may be readily worked out from the corresponding thermodynamic relations, while at the same time the equation shall represent a very close approximation to the truth over a *limited* range, the range chosen being one of practical importance. Whether such a formula can, or cannot, be extrapolated

* If in the (b) equation we put $2C = b$.

yond the limits of the range is a mattei of secondary interest—what important is that the formula should be as exact as may be within ese limits.

Let us then investigate the form which such an equation as the equation of Clausius assumes for moderate pressures. Re-iting the equation as

$$p(V - b) = RT - \frac{a(V - b)}{T(V + c)^2},$$

see that at modeiate pressures, when the volume is large, we shall t be seriously in error if we write

$$\frac{V - b}{(V + c)^2} = \frac{1}{V} \text{ (approximately).}$$

thus have

$$p(V - b) = RT - \frac{a}{TV};$$

l again, putting, in the small term,

$$V = \frac{RT}{p},$$

find, on rearranging the equation,

$$V = \frac{RT}{p} - \frac{c}{T^2} + b,$$

ere c is put for $\frac{a}{R}$.

If we replace T^2 by the more general form T^n, where n varies from stance to substance, we have

$$V = \frac{RT}{p} - \frac{c}{T^n} + b,$$

ch is the form known as Callendar's equation

This equation has been applied very successfully to elucidate properties of steam over a range of pressure from 0 to 34 atmos-res; the value of n appropriate to steam is $\frac{10}{3}$ Space will not mit us to discuss at length this important equation Indeed, the ussion lies within the province of thermodynamics, and the ler desirous of further information should consult the articles hermodynamics" and "Vaporization" in the *Encyclopædia 'annica*, or a textbook such as Ewing's *Thermodynamics for ineers*.

Osmotic Pressure

If we throw a handful of currants or raisins into water and leave them for a while, we find that the fruits, originally shrunken and wrinkled, have swelled out and become smooth.* Water has passed through the skin of the fruit, while the dissolved substances inside cannot pass out—or at least do not stream out so freely as the water streams in. This unilateral passage of a substance through a membrane is termed *osmosis*.†

In the limiting case, when we have a solution on one side of a membrane and pure solvent on the other, the membrane is called a semi-permeable membrane if it freely admits of the passage of the solvent, but is strictly impervious to the dissolved substance It has been asserted that no such membranes exist in nature, but, as far as experiment can show, a membrane of copper ferrocyanide forms a true semi-permeable membrane to a solution of sugar in water

Suppose such a membrane, prepared with due precautions ‡—and it is not so easy as one would imagine to prepare a thoroughly resistant membrane—to be deposited on the inside of a cylindrical porous pot. The pot is filled with a sugar solution, closed, and attached to a suitable manometer which shall measure the pressure inside the pot It is then placed in a vessel containing pure water We shall find that the pressure in the pot rises, finally reaching a maximum stable value The maximum value of this pressure, *assuming that the membrane is truly semi-permeable*, is called the osmotic pressure of the solution.

It will thus be seen that osmotic pressure is defined in terms of a semi-permeable membrane. It is only in very loose phraseology that one can speak of the osmotic pressure of a solution without reference to the existence of a semi-permeable membrane. A solution, *qua* solution, has no osmotic pressure.

If this definition be consistently followed, a great deal of vague and loose reasoning of the soporific-power-of-opium variety will be swept away. It is all the more needful to emphasize this point as there has arisen, in biological (and even in engineering) circles, a tendency to ascribe to " osmotic pressure " a power and potency

* Imbibition of water by the dried tissues will also play a part in the smoothing process

† From ὠσμός, a rare Greek noun meaning " thrusting " or " pushing through ".

‡ Morse, *Jour. Amer Chem. Soc.*, **45**, 91 (1911).

ich is almost proportional to the vagueness with which the chanism of that pressure is conceived.

Consider, for example, the common remark that osmotic pressure cts the wrong way "—that is, causes motion from a region of lower 1otic pressure to a region of higher osmotic pressure. It only uires a little consideration of the definition of osmotic pressure y to realize that the argument involves a ὕστερον πρότερον, it is clear that it is *osmosis* which produces osmotic pressure, not 1otic pressure which produces osmosis.

The quantitative laws of osmotic pressure were first studied by ffer, whose figures show that for dilute solutions the pressure, constant temperature, is proportional to the concentration, and at constant concentration the pressure is proportional to the olute temperature. We may therefore write

$$PV = KT,$$

it has been shown by van't Hoff that the constant K has the e value as the gas constant R Hence it follows that the osmotic ssure of the solution is the same as that which would be exerted the dissolved substance were it dispersed in the form of a gas ough a volume equal to that occupied by the solution.

If we desire to correct this simple gas law, we find it necessary ook at the matter from a different angle. We *define* an ideal tion as containing two completely miscible unassociated coments, of such a nature that there occurs no change of volume on ing, and that the heat of dilution is negligible

For such a solution it can be shown that the osmotic pressure P ven by

$$P + \beta\frac{P^2}{2} = \frac{RT}{V}\left\{-\log_e(1 - x)\right\},$$

re β denotes the compressibility of the solvent, V its molecular me, and x the ratio of the number of molecules of the dissolved stance to the total number present. If we neglect β, which is lly small, and expand the logarithmic term, we have the conent form

$$P = \frac{RT}{V}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right).$$

s equation holds good for any concentration.

Despite a large amount of criticism, the kinetic theory of osmotic sure still holds the field as the only one which gives values of the

pressure calculated from theory.* The properties of the membrane, which play a large part in some theories, whilst of great interest and value, are distinctly of secondary importance in the kinetic theory. It is the thermal agitation of the molecules of the solute which is effective in producing osmotic pressure, and the magnitude of the pressure calculated from the agitation of the molecules is equal to the value obtained by experiment. "Any other theory put forward to account for osmosis must fulfil, then, a double duty, not only must it be competent to explain osmosis, but it must also explain *away* the effects that we have the right to expect from the molecular agitation of the solute."†

NOTE ON TRANSFORMATION OF CO-ORDINATES

Perhaps the simplest way of passing from one form to the other is to consider the concentration of fluid, reckoned per unit volume per second, at a point

If u, v, w are the component velocities at a point P, the fluid leaving the elemental volume $\delta x \delta y \delta z$ in time δt is readily found, by the method given below for a more difficult case, to exceed that which enters it, by an amount

$$\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\delta x \delta y \delta z \delta t,$$

where ρ is the density, i.e the concentration, reckoned in mass per unit volume per second, is

$$-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Taking now the element of volume shown in fig. 16,

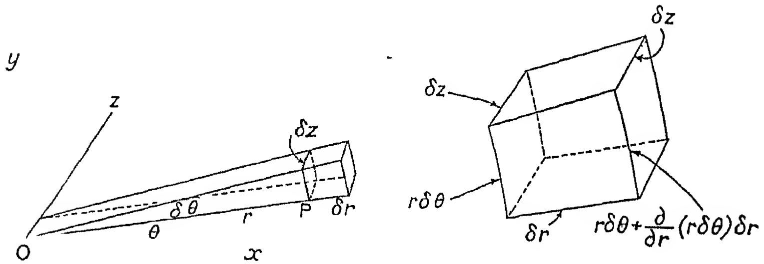


Fig. 16

* Porter, *Trans Far Soc*, **13**, 10 (1917) The reader desiring more information on the subject should consult this valuable discussion † Porter, loc. cit., p. 8.

ınd lettıng u' be the ıadial velocity and v' the tangentıal velocıty at P, ve get

$$u'r\delta\theta\delta z\delta t \text{ for the radıal flow in, and}$$

$$\left(u' + \frac{\partial u'}{\partial r}\delta r\right)\left(r\delta\theta + \frac{\partial}{\partial r}(r\delta\theta)\delta r\right)\delta z\delta t \text{ for the radial flow out.}$$

The latter exceeds the former by

$$\left(r\frac{\partial u'}{\partial r} + u'\right)\delta\theta\delta r\delta z\delta t.$$

The tangential flow in is $v'\delta r\delta z\delta t$,
nd the tangentıal flow out,

$$\left(v' + \frac{\partial v'}{\partial\theta}\delta\theta\right)\delta r\delta\theta\delta z\delta t,$$

ʼhıch exceeds the former expression by

$$\frac{\partial v'}{\partial\theta}\delta\theta\delta r\delta z\delta t,$$

e the total flow out ıs

$$\left(u' + r\frac{\partial u'}{\partial r} + \frac{\partial v'}{\partial\theta}\right)\delta\theta\delta r\delta z\delta t,$$

e the total flow *ınto* the element ıs thıs expıessıon taken wıth the ıınus sıgn

The elemental volume ıs $r\delta\theta\delta z\delta r$,
ence the concentratıon ıs

$$-\rho\left(\frac{u'}{r} + \frac{\partial u'}{\partial r} + \frac{1}{r}\frac{\partial v'}{\partial\theta}\right).$$

The concentratıon—mass per unıt volume per second—must be the ıme whatever co-ordınates we use, hence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{u'}{r} + \frac{\partial u'}{\partial r} + \frac{1}{r}\frac{\partial v'}{\partial\theta} \qquad .(1)$$

If the fluıd motıon has a velocıty potentıal, the component velocıty ı any dırection ıs the gradıent of potentıal ın that dırectıon, i e

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y},$$

$$u' = \frac{\partial\phi}{\partial r}, \quad v' = \frac{1}{r}\frac{\partial\phi}{\partial\theta},$$

for the element of length perpendicular to r, i.e. in the tangentia direction, is $rd\theta$; hence

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\equiv\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\phi}{\partial\theta^2}, \quad \ldots\ldots(2)$$

by substituting the values of u, u', v, v', in equation (1).

We have *proved* the transformation when the dependent variable is the velocity potential ϕ, but as the transformation is a purely analytical one in its nature, the form must be equally true whatever be the physical nature of ϕ; for instance, if $\phi\equiv w$, the z component of velocity,

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\equiv\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2},$$

the form which actually occurs (p. 33), and, in general,

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\equiv\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}, \quad \ldots\ldots\ldots(3)$$

at every point in a plane where U is a single-valued function of the co-ordinates of the point and possesses finite derivatives up to those of the second order there.

CHAPTER II

Mathematical Theory of Fluid Motion

It is assumed throughout this chapter that the fluid with which we deal may be regarded as incompressible. This only means that changes of pressure are propagated in it instantaneously, instead of with the (very great) velocity of sound. Since the velocity of sound in air is only about four times less than in water, it is clear that many of our results will be equally applicable to gaseous fluids, the influence of compressibility being negligible except in the case of very rapid differential motions.

It is further assumed in the first instance that the fluid is *frictionless*, i e. that it presses perpendicularly on any surface with which it is in contact, whether it be the surface of an adjacent portion of fluid, or of a solid boundary This hypothesis of the absence of all tangential stress is not in accordance with fact, but it greatly simplifies the mathematics of the subject, and there are, moreover, many cases of motion in which the influence of friction is only secondary It follows from this assumption that the state of stress at any point P of the fluid may be specified by a single quantity p, called the "pressure-intensity" or simply the "pressure", which measures the force per unit area exerted on any surface through P, whatever its aspect. This is in fact the cardinal proposition of hydrostatics.

It is convenient here to prove, once for all, that the resultant of the pressures exerted on the boundary of any small volume Q of fluid is a force whose component in the direction of any line element δs is

$$-\frac{\partial p}{\partial s}Q, \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

where $\partial p/\partial s$ is the gradient of p in the direction of δs. Take first the case of a columnar portion of fluid whose length δx is parallel to

he axis of x, and suppose that the dimensions of the cross-section ω) are small compared with δx. The pressure-intensities at the wo ends may then be lenoted by

$$p \text{ and } p + \frac{\partial p}{\partial x}\delta x,$$

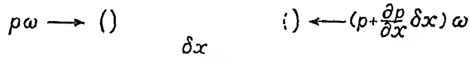


Fig 1

o that the component parallel to the length of the pressure on the olumn is

$$p\omega - \left(p + \frac{\partial p}{\partial x}\delta x\right)\omega = -\frac{\partial p}{\partial x}\omega\delta x,$$

he pressures on the sides being at right angles to δx Since $\omega\delta x$; the volume of the column, the formula (1) is in this case verified. ince, moreover, *any* small volume Q may be conceived as built up f columnar portions of the above kind, and since there is nothing pecial to the direction Ox, the result is seen to be general.

Stream-line Motion

1 *Bernoulli's Equation*

A state of *steady* or *stream-line* motion is one in which the stream-nes, i e the actual paths of the particles, preserve their configuration nchanged The most obvious examples are where a stream flows ast a stationary solid, and the designation is naturally extended) cases where a solid moves uniformly in a straight line, without otation, through a surrounding fluid, provided the superposition of uniform velocity equal and opposite to that of the solid reduces ie case to one of steady motion in the former sense This super-osed velocity does not of course make any difference to the dynamics f the question

The stream-lines drawn through the contour of any small area ill mark out a tube, which we may call a *stream-tube*. Since the ime volume of fluid must traverse each cross-section in the same me, we have

$$q_1\omega_1 = q_2\omega_2,$$

here ω_1, ω_2 are the areas of any two cross-sections at P_1, P_2, and q_1, q_2 the corresponding velocities in the direction (say) from P_1) P_2. Now consider the region included between these two sections. 1 a short time dt a volume $Q = \omega_1 q_1 dt$ will have entered it at

P_1, and an equal volume $Q = \omega_2 q_2 dt$ will have left it at P_2 The work done by hydrostatic pressure in this time on the mass of fluid which originally occupied the space P_1P_2 will be $p_1\omega_1 q_1 dt$ or p_1Q_1, at P_1, and $-p_2\omega_2 q_2 dt$ or $-p_2Q$ at P_2. The same mass will have gained kinetic energy of amount $\frac{1}{2}\rho Q(q_2{}^2 - q_1{}^2)$, where ρ is the density, i.e. the mass per unit volume. If V denotes the potential energy of unit mass, the gain of potential energy will be $\rho Q(V_2 - V_1)$ Hence, equating the work done on the mass to the total increment of energy we have

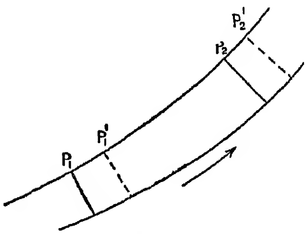


Fig. 2

$$p_1 - p_2 = \tfrac{1}{2}\rho(q_2{}^2 - q_1{}^2) + \rho(V_2 - V_1),$$

or

$$p_1 + \tfrac{1}{2}\rho q_1{}^2 + \rho V_1 = p_2 + \tfrac{1}{2}\rho q_2{}^2 + \rho V_2$$

Hence along any stream-line

$$p + \tfrac{1}{2}\rho q^2 + \rho V = C, \qquad (2)$$

where C is a constant for that particular line, but may vary from one stream-line to another. This equation is due to D. Bernoulli (1738) and was proved by him substantially in the above manner

The formula has many applications For instance, in the case of water issuing from a small orifice in the wall of an open vessel we have at the upper surface $p = p_0$ (the atmospheric pressure), and $q = 0$, approximately Again, the value of V at the upper surface exceeds that at the orifice by gz, where z is the difference of level and g the acceleration due to gravity Hence if q be the velocity at the surface of the issuing jet,

$$p_0 + g\rho z = p_0 + \tfrac{1}{2}\rho q^2,$$

or

$$q^2 = 2gz, \qquad (3)$$

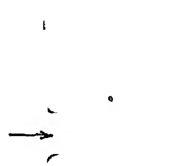


Fig 3

a formula due to Torricelli (1643). If S′ be the section of the jet at the "vena contracta", where it is sensibly parallel, the pressure over S′ will be p_0 The velocity will therefore be given by the above value of q, and the discharge per unit time will be $\rho q S'$ The ratio of S′ to the area S of the orifice is called the "coefficient of contraction". It is not easy to determine this coefficient theoretically, but a

'ery simple argument shows that in the case of an orifice in a hin wall it must exceed $\frac{1}{2}$. Take, for instance, an orifice in a 'ertical wall. In every second a mass of $\rho q S'$ escapes with the elocity q, and carries with it a momentum $\rho q^2 S'$. This represents he horizontal force exerted by the vessel on the fluid. There must e a contrary reaction of this amount on the vessel. On the opposite vall of the vessel, where the velocity is nsignificant, the pressure has sensibly the tatical value due to the depth, and if this vere also the case on the wall containing he orifice there would be an unbalanced orce $g\rho z S$ urging the vessel backwards ctually, owing to the appreciable elocity, the pressure near the orifice will e somewhat less, so that the reaction exeeds $g\rho z S$. Hence $\rho q^2 S' > g\rho z S$, or (since $^2 = 2gz$) $S' > \frac{1}{2}S$ In a particular case, where the fluid escapes y a tube projecting inwards in the manner shown in the figure, the atical pressure obtains practically over the walls, and $S' = \frac{1}{2}S$ xactly. This arrangement is known as "Borda's mouthpiece"

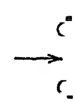


Fig 4

Another application of (3), much used in aeronautics and engieering, is to the measurement of the velocity of a stream, e g of ie relative wind in an eroplane. The quantities and $p + \frac{1}{2}\rho q^2$ are meaured independently, and ieir difference determines A fine tube, closed at ie end and connected with pressure-gauge at the her, points up the stream as to interfere as little as possible with the motion, and contains few minute holes in its side, at a little distance from the closed id; the gauge therefore gives the value of p On the other hand, open tube drawn out at the end almost to a point, and nnected to a second gauge, will give the value of $p + \frac{1}{2}\rho q^2$ at a ort distance ahead of the vertex. For if p' be the pressure at the rtex itself, where the velocity is arrested, we have $p + \frac{1}{2}\rho q^2$ p', for points on the same stream-line. The two contrivances e often united (as in the figure) in a single appliance known as a Pitot and static pressure tube".

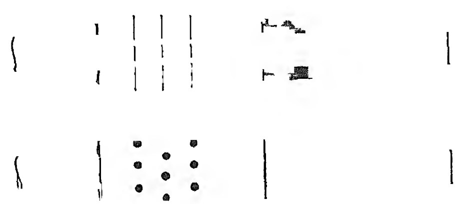


Fig 5

2. *Two-dimensional Motion. Stream-function*

There are two types of stream-line motion which are specially simple and important. We take first the two-dimensional type, where the motion is in a system of parallel planes, and the velocity has the same magnitude and direction at all points of any common normal. It is sufficient then to confine our attention to what takes place in one of these planes. Any line drawn in it may be taken to represent the portion of the cylindrical surface, of which it is a cross-section, included between this plane and a parallel plane at unit distance from it. By the "flux" across the line we understand the volume of fluid which in unit time crosses the surface thus defined Now taking an arbitrarily fixed point A and a variable point P, the flux (say from right to left) will be the same across any two lines drawn from A to P, provided the space between them is wholly occupied by fluid This is in virtue of the assumed constancy of volume The flux will therefore be a function only of the position of P, it is usually denoted by the letter ψ. It is evident at once from the definition that the value of ψ will not alter as the point P describes a stream-line, and therefore that the equation

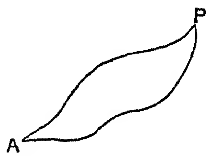


Fig 6

$$\psi = \text{constant} \qquad (4)$$

will define a stream-line For this reason ψ is called the *stream-function*. If P′ be any point adjacent to P, the flux across AP′ will differ from that across AP by the flux across PP′, whence, writing $PP' = \delta s$, we have $\delta\psi = q_n \delta s$, where q_n is the component velocity normal to PP′, to the left. Thus

$$q_n = \frac{\partial\psi}{\partial s}, \qquad (5)$$

where $\partial\psi/\partial s$ is the gradient of ψ in the direction of δs. This leads to expressions for the component velocities u, v parallel to rectangular co-ordinate axes. If we take δs parallel to Oy we have $q_n = -u$, whilst if it be taken parallel to Ox we have $q_n = v$. Thus

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x} \qquad (6)$$

These satisfy the relation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (7)$$

which is called the *equation of continuity*. It may be derived otherwise by expressing that the total flux across the boundary of an elementary area $\delta x \delta y$ is zero. Again, if we use polar co-ordinates r, θ, and take δs $(= \delta r)$ along the radius vector, we have $q_n = v$, where v denotes the transverse velocity; whilst if δs $(= r\delta\theta)$ be at right angles to r, $q_n = -u$, the radial velocity. Hence

$$u = -\frac{\partial\psi}{r\partial\theta}, \quad v = \frac{\partial\psi}{\partial r}. \quad \ldots\ldots\ldots\ldots\ldots(8)$$

It follows that

$$\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial\theta} = 0, \quad \ldots\ldots\ldots\ldots\ldots(9)$$

which is another form of the equation of continuity.

It is to be remarked that the above definition of ψ is purely geometrical, and is merely a consequence of the assumed incompressibility of the fluid. If we make any assumption whatever as to the form of this function, the formulæ (6) or (8) will give us a possible type of motion; but it by no means follows that it will be a possible type of permanent or steady motion. To ascertain the condition which must be fulfilled in order that this may be the case we must have recourse to dynamics, but before doing this it is convenient to introduce the notions of *circulation* and *vorticity*.

The *circulation* round any closed line, or circuit, in the fluid is the line-integral of the tangential velocity taken round the curve in a prescribed sense. In symbols it is

$$\int q \cos\chi \, ds, \quad \ldots(10)$$

where χ is the angle which the direction of the velocity q makes with that of the line-element δs. In rectangular co-ordinates, resolving u and v in the direction of δs, we have

$$q \cos\chi = u\frac{dx}{ds} + v\frac{dy}{ds},$$

so that the circulation is

$$\int\left(u\frac{dx}{ds} + v\frac{dy}{ds}\right)ds, \text{ or } \int(udx + vdy) \quad \ldots\ldots(11)$$

It will appear that the circulation round the contour of an infinitely small area is ultimately proportional to the area. The ratio which it bears to the area measures (in the present two-dimensional case) the *vorticity*; we denote it by ζ. Its value, in terms of rectangular

co-ordinates, is found by calculating the circulation round an elementary rectangle PQRS whose sides are δx and δy. The portions of the line-integral (11) due to PQ and RS are together equal to the difference in the corresponding values of $u\delta x$, i.e. to $-\frac{\partial u}{\partial y}\delta y \delta x$. The portions due to QR and SP are in like manner equal to $\frac{\partial v}{\partial x}\delta x \delta y$. Equating the sum to $\zeta \delta x \delta y$, we have

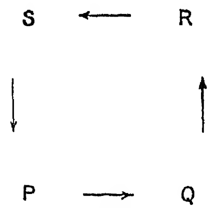


Fig. 7

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

or, from (6),

$$\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

The value of ζ at any point P is related to the average rotation relative to P of the particles in the immediate neighbourhood To examine this, we calculate the circulation round a circle of small radius r having P as centre The velocity at any point Q on the circumference may be regarded as made up of a general velocity equal to that of P, and the velocity relative to P. The former of these contributes nothing to the required circulation The latter gives a tangential component ωr, where ω is the angular velocity of QP The circulation is therefore

$$\int_0^{2\pi} \omega r \, r d\theta = r^2 \int_0^{2\pi} \omega d\theta,$$

where θ is the angular co-ordinate of Q. Since the same thing is expressed by $\zeta \pi r^2$, we have

$$\zeta = \frac{1}{\pi} \int_0^{2\pi} \omega d\theta, \quad \ldots\ldots\ldots\ldots (14)$$

which is *twice* the average value of ω on the circumference. For this reason a type of motion in which ζ is everywhere zero, i.e in which the ratio of the circulation in every *infinitesimal* circuit to the area within the circuit vanishes, is called *irrotational.*

3. *Condition for Steady Motion*

We can now ascertain the dynamical conditions which must e satisfied in order that a given state of motion may be steady. 'or this purpose we consider the forces acting on an element, 'QQ'P', of fluid included be-ween two adjacent stream-lines nd two adjacent normals. The tter meet at the centre of cur-ature C. We write PQ $= \delta s$, P' $= \delta n$, PC $=$ R. The mass f the element is therefore $\rho\delta s\delta n$. 'he forces acting on it may be solved in the direction of the ngent and normal, respectively, the stream-line PQ. Tan-ntial resolution would merely ad us again, after integration, Bernoulli's equation (2). ormal resolution gives, with e help of (1),

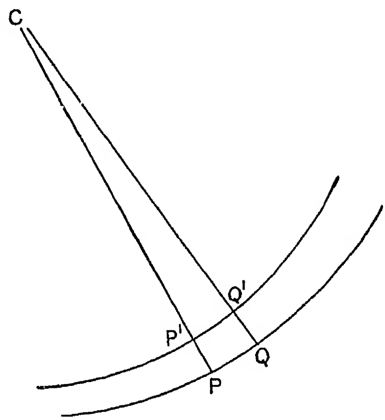


Fig 8

$$\rho\delta s\delta n\frac{q^2}{R} = -\frac{\partial p}{\partial n}\delta s\delta n - \frac{\partial V}{\partial n}\rho\delta s\delta n,$$

here $\partial p/\partial n$ and $\partial V/\partial n$ are the gradients of p and V in the direction C Hence

$$\frac{\partial}{\partial n}(p + \rho V) = -\frac{\rho q^2}{R}. \quad \ldots\ldots\ldots \quad (15)$$

he circulation round the circuit PQQ'P' will be equal to $\zeta\delta s\delta n$ calculating this circulation, we may neglect the sides PP', QQ' ice they are at right angles to the velocity The contributions of e remaining sides are

$$q\delta s \quad \text{and} \quad -\left(q + \frac{\partial q}{\partial n}\delta n\right)\delta s',$$

iere $\delta s' =$ P'Q'. Now from the figure we have

$$\frac{\delta s'}{\delta s} = \frac{CP'}{CP} = \frac{R - \delta n}{R},$$

that the circulation is

$$q\delta s - \left(q + \frac{\partial q}{\partial n}\delta n\right)\left(1 - \frac{\delta n}{R}\right)\delta s = \left(\frac{q}{R} - \frac{\partial q}{\partial n}\right)\delta s\delta n,$$

omitting terms of higher order than those retained Hence

$$\zeta = -\frac{\partial q}{\partial n} + \frac{q}{R} \quad \ldots\ldots\ldots\ldots (16)$$

The formula (15) may now be written

$$\frac{\partial}{\partial n}(p + \tfrac{1}{2}\rho q^2 + \rho V) = -\rho q\zeta \quad \ldots\ldots (17)$$

Comparing this with (2), we have

$$\delta C = -\rho q\zeta\delta n, \quad \ldots\ldots (18)$$

where C is the quantity which was proved before to be constant along any stream-line, but will in general vary from one stream-line to another. If we fix our attention on two consecutive stream-lines, δC will be a constant, and $q\delta n$ will also obviously be constant The dynamical condition for steady motion is therefore that the vorticity should be constant along any stream-line. When it is fulfilled, the distribution of pressure is given by (2) and (17) We may express the result otherwise by saying that any fluid element retains its vorticity unchanged as it moves along. This is a particular case of a theorem in vortex-motion to be proved later

An obvious example is that of fluid rotating with uniform angular velocity ω about a vertical axis, and subject to gravity The law of distribution of pressure may be deduced from (17), or more simply from first principles. If r be the radius of the circular path of a small volume Q, the resultant force upon it must be radial, of amount $\rho Q\omega^2 r$. Hence, and since there is no vertical acceleration, we have

$$\rho\omega^2 r = \frac{\partial p}{\partial r}, \quad 0 = -\frac{\partial p}{\partial z} - g\rho, \quad \ldots\ldots (19)$$

the positive direction of z being that of the upward vertical. It follows that

$$p = \tfrac{1}{2}\rho\omega^2 r^2 - g\rho z + \text{constant.} \quad \ldots\ldots (20)$$

The free surface ($p = p_0$) is therefore the parabola

$$z = \frac{\omega^2}{2g} r^2. \quad \ldots\ldots\ldots\ldots (21)$$

if the origin of z is where the free surface meets the axis ($r = 0$).

If we imagine the fluid contained within the cylindrical surface $= a$, rotating in the above manner, to be surrounded by fluid ioving irrotationally, we have in the latter region $\partial q/\partial r + q/r = 0$, om (16), or

$$qr = \text{constant} = \omega a^2 \qquad (22)$$

ernoulli's equation then gives

$$p = \text{constant} - g\rho z - \tfrac{1}{2}\rho\frac{\omega^2 a^4}{r^2} \qquad (23)$$

he equation to the free surface is therefore

$$z = -\frac{\omega^2 a^4}{2gr^2} + \frac{\omega^2 a^2}{g}, \qquad (24)$$

here the additive constant has been chosen so as to agree with 1) when $r = a$ It appears that these equations also give the

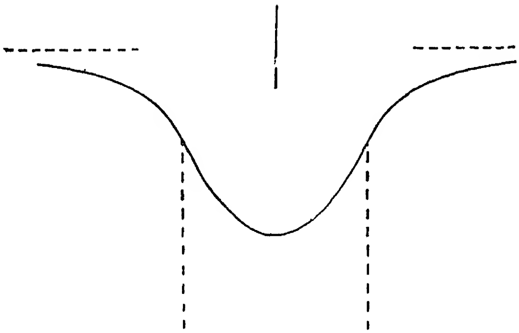


Fig 9

ne value of dz/dr for $r = a$ Putting $r = \infty$ in (24), we find it the depth of the dimple formed on the free surface is $\omega^2 a^2/g$.

4. *Irrotational Motion*

We proceed now to consider more particularly the case of irro- ional motion. The condition for steady motion is fulfilled auto- itically if $\zeta = 0$ everywhere, provided, of course, the necessary undary conditions are satisfied, as they are in the case of the w of a liquid past a stationary solid. The geometrical condition ;) reduces to

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \qquad (25)$$

or, in polar co-ordinates,

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\delta r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0 \qquad \ldots\ldots\ldots\ldots(26)$$

and the pressure-distribution is such that

$$p+\tfrac{1}{2}\rho q^2+\rho V = \text{constant} \qquad \ldots\ldots(27)$$

throughout the fluid. The particular value of the constant is for most purposes unimportant, since the addition of a uniform pressure throughout does not alter the resultant force on any small element of fluid, or on an immersed solid.

Some simple solutions of (25) or (26) are easily obtained. Thus

$$\psi = -Uy = -Ur\sin\theta \qquad \ldots\ldots(28)$$

gives a uniform flow with velocity U from left to right. Again, take the case of symmetrical radial flow outwards from the origin. The stream-lines are evidently the radii, so that ψ is a function of θ only. Since the total flux outwards across any circle r = constant must be the same, we have from (8)

$$-\frac{\partial\psi}{r\partial\theta}.2\pi r = \text{constant} = m, \text{ say,}$$

$$\text{or} \qquad \psi = -\frac{m}{2\pi}\theta. \qquad \ldots\ldots(29)$$

If m be positive we have here the fictitious conception of a *line-source* which emits fluid at a given rate If m be negative we have a *sink* Since (29) would make the velocity infinite at the origin, these imaginary sources and sinks must be external to the region occupied by the fluid. The formula (29), for instance, would be realized by the expansion of a circular cylinder whose axis passes through O. Again, since the differential equations (25) and (26) are linear, they are satisfied by the sum of any number of separate solutions. For instance, the combination of a source at A, and an equal sink at a point B to the left of A, gives

$$\psi = -\frac{m}{2\pi}(\theta_1-\theta_2), \qquad \ldots\ldots\ldots\ldots(30)$$

where θ_1, θ_2 are the angles which the lines drawn from A and B to any point P make with the direction BA. Since $\theta_1-\theta_2$ = APB, the lines ψ = constant are a system of circles through A, B. This

ind of motion would involve infinite velocities at A and B, but if e combine (28) with (30) we get the flow past an oval cylinder hich encloses the imaginary source and sink. If the points A and be made to approach one another, whilst m increases so that the roduct mAB is constant, we have ultimately $\theta_1 - \theta_2 = \text{APB} =$ B $\sin\theta/r$. We thus get the form

$$\psi = \mathrm{C}\frac{\sin\theta}{r} \quad \ldots (31)$$

ombining this with (28), we have

$$\psi = \left(-\mathrm{U}r + \frac{\mathrm{C}}{r}\right)\sin\theta \quad \ldots (32)$$

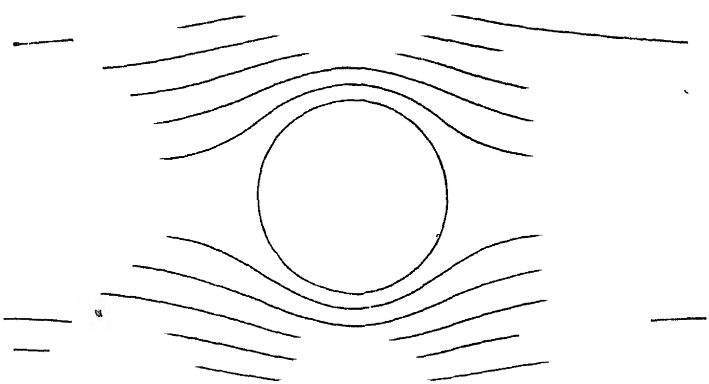


Fig 10

e stream-line $\psi = 0$ now consists of the radii $\theta = 0$, $\theta = \pi$, d the circle $r = a$, provided $\mathrm{C} = \mathrm{U}a^2$. The formula

$$\psi = -\mathrm{U}\left(r - \frac{a^2}{r}\right)\sin\theta \quad \ldots (33)$$

erefore gives the flow past a circular cylinder. The normal velocity the surface is of course zero, whilst the tangential velocity is

$$\frac{\partial\psi}{\partial r} = -\mathrm{U}\left(1 + \frac{a^2}{r^2}\right)\sin\theta = -2\mathrm{U}\sin\theta \quad \ldots (34)$$

nitting the external forces, if any, represented by V, which have rely an effect analogous to buoyancy, the pressure at the cylinder

$$p = \text{constant} - 2\rho \mathrm{U}^2\sin^2\theta \quad \ldots (35)$$

Since this is unaltered when θ is replaced by $-\theta$, or by $\pm(\pi-\theta)$, it is evident that the stream exerts no resultant force on the cylinder. Some qualification of this result will be given presently. Meantime we note that if we superpose a general velocity U from right to left, we get the case of a cylinder moving with uniform velocity (and zero resistance) through a fluid which is at rest at infinity. The stream-function then has the form

$$\psi = \frac{Ua^2}{r}\sin\theta, \qquad (36)$$

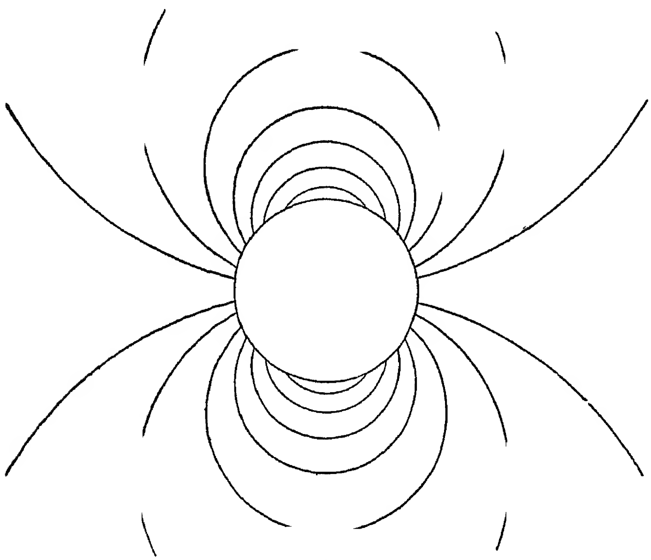


Fig 11

so that the relative stream-lines are portions of the circles $r = C\sin\theta$, which touch the axis of x at the origin. If we calculate the square of the velocity from (36), we find

$$q^2 = \left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{r\partial\theta}\right)^2 = \frac{U^2a^4}{r^4}.$$

The total kinetic energy of the fluid is therefore

$$\int_a^\infty \tfrac{1}{2}\rho q^2 \cdot 2\pi r dr = \tfrac{1}{2}\pi\rho U^2 a^2 = \tfrac{1}{2}M'U^2, \ldots\ldots\ldots (37)$$

where M′ is the mass of fluid displaced by the cylinder. The effect

f the presence of the fluid is therefore virtually to increase the ıertia of the latter by M′.

Another sımple type of motion ıs where the fluid moves ın ɔncentric circles about O. The velocity is then a function of r ıly If the motıon is irrotational we must have, by (22),

$$qr = \text{constant} = \kappa/2\pi,$$

$$\text{or} \qquad r\frac{\partial\psi}{\partial r} = \frac{\kappa}{2\pi},$$

ıere κ is the constant value of the cırculatıon round O. Thus

$$\psi = \frac{\kappa}{2\pi}\log r, \qquad \ldots\ldots\ldots\ldots(38)$$

additıve constant beıng without effect. Thıs corresponds to the ɜe of a concentrated *lıne-vortex* at the orıgin, and would give inite velocity there. For this ıeason (38) can only relate to cases ıere the orıgın is external to the space occupied by the fluıd

The combınatıon of (33) and (38) makes

$$\psi = -\,\mathrm{U}\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\kappa}{2\pi}\log r \qquad \ldots\ldots(39)$$

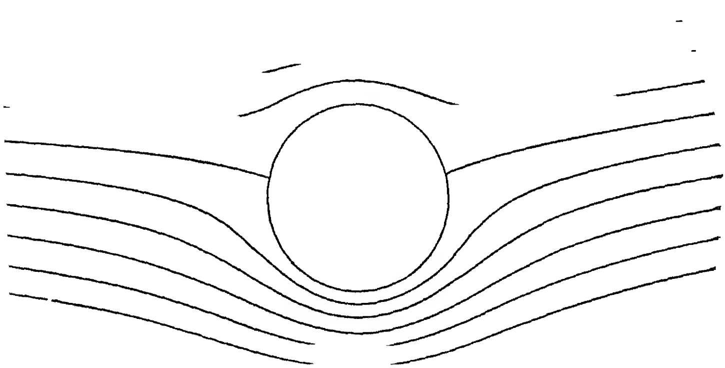


Fig 12

: tangential velocity at the cylinder is now

$$\frac{\partial\psi}{\partial r} = -\,2\mathrm{U}\sin\theta + \frac{\kappa}{2\pi a},$$

whence

$$p = \text{constant} - 2\rho U^2 \sin^2\theta + \rho\frac{\kappa U}{\pi a}\sin\theta \ldots\ldots \quad (40)$$

The last term is the only one which contributes to a resultant force. Since it is the same for θ and $\pi - \theta$, there is on the whole no force parallel to Ox. The force parallel to Oy is

$$-\int_0^{2\pi} p \sin\theta a d\theta = -\frac{\rho k U}{\pi}\int_0^{2\pi}\sin^2\theta d\theta = -\rho\kappa U \ldots (41)$$

This resultant effect is due to the fact that (if κ be positive) the circulation diminishes the velocity above the cylinder and increases it below, and that a smaller velocity implies (other things being the same) a greater pressure. It may be shown that the result is the same for a cylinder of any form of section, as might be expected from the fact that it does not depend on the radius a. This theorem is the basis of Prandtl's theory of the lift of an aeroplane.

5. *Velocity-potential*

We may imagine any area occupied by fluid to be divided by a double series of lines crossing it into infinitesimal elements. The circulation round the boundary of the area will be equal to the sum of the circulations round the various elements, provided these circulations be estimated in a consistent sense. For, in this sum, a

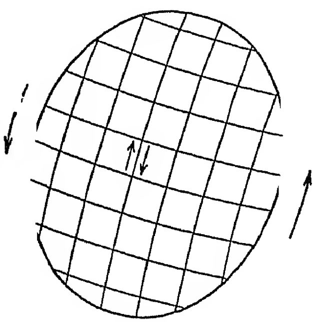


Fig 13

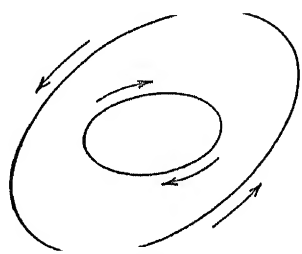


Fig 14

side common to two adjacent elements contributes amounts which cancel. Hence if the motion be irrotational the circulation round the boundary of any area wholly occupied by fluid will be zero. We have here assumed the boundary to consist of a single closed

urve. If it consists of two such curves, what is proved is that the um of the circulations round these in opposite senses is zero. In ther words, in irrotational motion the circulation in the *same* sense ; the same for any two circuits which can by continuous modifica-ion be made to coincide without passing out of the region occupied y the fluid. For example, in the case to which (38) refers, the cir-ulation in any circuit which embraces the cylinder is κ, whilst that 1 any other circuit is zero.

This leads to the introduction of the function called the *velocity-otential*, in terms of which problems of irrotational motion are often iscussed. This is defined by the integral

$$\phi = -\int_A^P (udx + vdy). \qquad \ldots\ldots\ldots(42)$$

ken along a line drawn from A to P. The integral has the same ılue for any two such lines, such as ABP, ACP in the figure, pro-ded the space between them is fully :cupied by fluid. For, reversing the rection of one of these lines, the ıths ABP, PCA together form a osed circuit, round which the circu-tion is zero It follows that so long A is fixed, ϕ will be a function of e position of P only. If P′ be any ›int adjacent to P, the increment of in passing from P to P′ is $\delta\phi = -q_s\delta s$, where q_s is the component locity in the direction PP′, and PP′ $= \delta s$. Hence

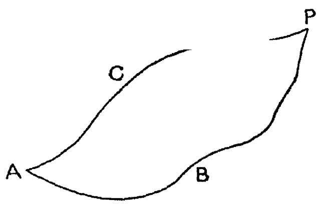


Fig 15

$$q_s = -\frac{\partial\phi}{\partial s}, \qquad (43)$$

ıere $\partial\phi/\partial s$ is the gradient of ϕ in the direction PP′. For instance, rectangular co-ordinates, putting first $\delta s = \delta x$, and then $= \delta y$, · have

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y} \qquad \ldots\ldots\ldots(44)$$

nilarly, the radial and transverse velocities in polar co-ordinates : given by

$$u = -\frac{\partial\phi}{\partial r}, \quad v = -\frac{\partial\phi}{r\partial\theta} \qquad \ldots\ldots\ldots(45)$$

From (7) and (44) we deduce

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \qquad (46)$$

whilst in polar co-ordinates, from (9) and (45)

$$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} = 0. \qquad (47)$$

It is the similarity between these relations and those met with in the theories of attractions and electrostatics that has suggested the name "velocity-potential". For the same reason the curves for which ϕ is constant are called *equipotential lines* If in (43) δs be taken along such a line we have $q_s = 0$, showing that the equipotential lines cut the stream-lines at right angles If on the other hand δn be the perpendicular distance between two adjacent equipotential lines, we have $\delta\phi = -q\delta n$ If, therefore, we imagine a whole system of such lines to be drawn for equal small increments $\delta\phi$, the perpendicular distance between consecutive lines will be everywhere inversely proportional to the velocity. If, further, we suppose the stream-lines to be drawn for intervals $\delta\psi$ each equal to $\delta\phi$, we have $\delta\psi = q\delta s'$, where $\delta s'$ is the interval between consecutive stream-lines of the system. Hence $\delta s' = \delta n$, showing that the stream-lines and equipotential lines drawn for equal increments of the functions will divide the region occupied by the fluid into infinitesimal squares.

The functions ϕ and ψ are connected by the relations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}, \qquad (48)$$

in which the equations (25) and (46), expressing the incompressibility and the absence of vorticity, are implied. If we write

$$w = \phi + i\psi, \quad z = x + iy, \qquad (49)$$

where $i = \sqrt{(-1)}$, the relations (48) are satisfied by any assumption of the form

$$w = f(z). \qquad (50)$$

For this makes

$$\frac{\partial w}{\partial y} = if'(z) = i\frac{\partial w}{\partial x}, \qquad (51)$$

whence, substituting the value of w, and equating separately real and imaginary parts, we reproduce (48).

For example, if $w = -\mathrm{U}z$, we have

$$\phi = -\mathrm{U}x, \quad \psi = -\mathrm{U}y, \qquad \text{.............(52)}$$

expressing a uniform flow parallel to Ox. Again, if $w = \mathrm{C}/z$

$$\phi + i\psi = \frac{\mathrm{C}}{x + iy} = \frac{\mathrm{C}}{r}(\cos\theta - i\sin\theta) \qquad \text{..... (53)}$$

This corresponds to (36), if $\mathrm{C} = -\mathrm{U}a^2$, and shows that in the case referred to

$$\phi = -\frac{\mathrm{U}a^2}{r}\cos\theta \qquad \text{............... (54)}$$

A more general assumption is

$$w = \mathrm{C}z^n,$$

or $$\phi + i\psi = \mathrm{C}(x + iy)^n = \mathrm{C}r^n(\cos n\theta + i\sin n\theta).$$

The stream-function is now

$$\psi = \mathrm{C}r^n \sin n\theta,$$

which vanishes both for $\theta = 0$ and $\theta = \alpha$, provided $n = \pi/\alpha$. Taking these lines as fixed boundaries we have the flow in an angle, or round a salient, according as $\alpha \lesseqgtr \pi$. The radial and transverse velocities are, by (8),

$$-n\mathrm{C}r^{n-1}\cos n\theta \quad \text{and} \quad n\mathrm{C}r^{n-1}\sin n\theta,$$

respectively If $\alpha < \pi, n > 1$, and these expressions vanish at the vertex where $r = 0$ If, on the other hand $\alpha > \pi$, $n < 1$, and the velocity there is infinite Even if the salient be rounded off, the velocity may be very great, with the result that the pressure falls much below the value at a distance It is otherwise obvious that if the fluid is to be guided round a sharp curve there must be a rapid increase of pressure outwards to balance the centrifugal force If this is not sufficient a vacuum is formed and "cavitation" ensues.

If $w = \mathrm{C}\log z$, where C is real,

$$\phi + i\psi = \mathrm{C}\log(x + iy) = \mathrm{C}\log re^{i\theta} = \mathrm{C}\log r + i\mathrm{C}\theta. \qquad (55)$$

This represents a line-source of strength m, if to agree with (29) we put $\mathrm{C} = -m/2\pi$. The corresponding value of ϕ is

$$\phi = -\frac{m}{2\pi}\log r. \qquad \text{...............(56)}$$

If on the other hand C is a pure imaginary, $= i\text{A}$, say,

$$\phi + i\psi = -\text{A}\theta + i\text{A}\log r \quad \ldots \quad (57)$$

This represents the case of the line-vortex to which (38) refers, if we put $\text{A} = \kappa/2\pi$, and so make

$$\phi = -\frac{\kappa}{2\pi}\theta \quad \ldots \quad (58)$$

The function ϕ has an important dynamical interpretation. Any state of motion in which there is no circulation in any circuit, and in which, therefore, ϕ has a definite value at every point, could be generated instantaneously from rest by a proper application of impulsive pressures over the boundary. For the requisite condition for this is that the resultant of the impulsive pressures (ϖ) on the surface of any small volume Q should be equivalent to the momentum acquired by this. Hence if q_s is the component velocity in the direction of any linear element δs we must have

$$-\frac{\partial \varpi}{\partial s}\text{Q} = \rho\text{Q}q_s = -\rho\text{Q}\frac{\partial \phi}{\partial s},$$

which is satisfied if

$$\varpi = \rho\phi \quad \ldots \quad (59)$$

Hence ϕ determines the impulsive pressure requisite to start the actual motion in the above manner.

As an example, we may take the case of a cylinder moving through a large mass of liquid, without circulation, to which the formula (54) refers. The resultant of the impulsive pressures on the surface of the cylinder is parallel to Ox, of amount

$$-\int_0^{2\pi} \omega \cos\theta \, a d\theta = \rho\text{U}a^2\int_0^{2\pi} \cos^2\theta \, d\theta = \text{M}'\text{U}, \quad \ldots (60)$$

if $\text{M}' = \pi\rho a^2$ as before. The total impulse which must be given to the *cylinder* to start the motion is therefore $(\text{M} + \text{M}')\text{U}$. This confirms the former result that the inertia of the cylinder is virtually increased by the amount M′.

6. *Motion with Axial Symmetry. Sources and Sinks*

The second type of motion to which reference was made on 54 is where the flow takes place in a system of planes passing rough an axis, which we take as axis of x, and is the same in each ch plane. We denote by x, y the co-ordinates in one of these ines, by r distance from the origin, and by θ the angle which r akes with Ox. The conditions for steady motion are obtained by e previous process. Resolving along a stream-line we should be l to Bernoulli's equation (2); whilst the normal resolution in an ial plane yields equations of the same form as (15) and (17), pro-ded ζ now denotes the vorticity in that plane The inference to the distribution of vorticity is however altered. The space tween two consecutive stream-lines now represents a section of thin shell, of revolution about Ox, and the flux in this is accord-gly $q\, 2\pi y \delta n$ Comparing with (18), we see that along any stream-e ζ must vary as y. We may conceive the fluid as made up of nular filaments having Ox as a common axis. The section of such filament, as it moves along, will vary inversely as y, hence the oduct of the vorticity into the cross-section must remain constant. his is a particular case of a general theorem that the *strength* of a *rtex-filament* (in this case a *vortex-ring*) remains unaltered as it oves

If $\zeta = 0$, the argument for the existence of a velocity-potential ll hold as before One or two simple cases may be noticed If imagine a *point-source* at O, the flux outwards across any con-ntric spherical surface of radius r must be equal to the output) per unit time whence

$$-\frac{\partial\phi}{\partial r}4\pi r^2 = m, \text{ or } \phi = \frac{m}{4\pi}\frac{1}{r}. \quad . \quad . \quad . \qquad .(61)$$

We may apply this solution to the collapse of a spherical bubble. R be the radius at time t, we have

$$\phi = \frac{R^2}{r}\frac{dR}{dt}, \qquad . \qquad .. \quad (62)$$

nce this makes $-\partial\phi/\partial r = dR/dt$ for $r = R$. The corresponding netic energy of the fluid is

$$\int_R^\infty \tfrac{1}{2}\rho q^2.4\pi r^2 dr = 2\pi\rho R^4\left(\frac{dR}{dt}\right)^2\int_R^\infty\frac{dr}{r^2} = 2\pi\rho R^3\left(\frac{dR}{dt}\right)^2. \quad . \quad ..(63)$$

If p_0 be the pressure at a distance, the rate at which work is being done on the fluid enclosed in a spherical surface of large radius r is

$$-p_0 q \cdot 4\pi r^2 = -4\pi p_0 R^2 \frac{dR}{dt}, \qquad \text{(64)}$$

the pressure inside the bubble being neglected. Equating the rate of increase of the energy to the work done,

$$\frac{d}{dt} R^3 \left(\frac{dR}{dt}\right)^2 = -\frac{2p_0}{\rho} R^2 \frac{dR}{dt}, \qquad \text{(65)}$$

whence

$$R^3 \left(\frac{dR}{dt}\right)^2 = \frac{2}{3}\frac{p_0}{\rho}(R_0^3 - R^3), \qquad \text{(66)}$$

where R_0 is the initial radius of the cavity It is not easy to integrate this further in a practical form, but the time of collapse happens to be ascertainable, it is

$$\tau = 0{\cdot}915 R_0 \sqrt{\frac{\rho}{p_0}} \qquad \text{(67)}$$

Thus if p_0 be the atmospheric pressure, and $R_0 = 1$ cm., τ is less than the thousandth part of a second. The total energy lost, or rather converted into other forms is, from (63) and (66),

$$\frac{4}{3}\pi p_0 R_0^3 \qquad \text{(68)}$$

In the particular case referred to, this is $4\ 19 \times 10^6$ ergs, or $0\ 312$ of a foot-pound

The expansion of a spherical cavity owing to the pressure of an included gas can be treated in a similar way This illustrates, at all events qualitatively, the early stages of a submarine explosion The potential energy of a gas compressed under the adiabatic condition to volume v and pressure p is $pv/(\gamma - 1)$, where γ is the ratio of the two specific heats. If p be the internal pressure when the radius of the cavity is R, and p_0 its initial value, we have by the adiabatic law

$$\frac{p}{p_0} = \left(\frac{R_0}{R}\right)^{3\gamma} \qquad \text{(69)}$$

The potential energy is therefore

$$\frac{4p_0 R_0^3}{3(\gamma - 1)}\left(\frac{R_0}{R}\right)^{3\gamma - 3} \qquad \text{(70)}$$

xpressing that the total energy is constant, we have

$$2\pi\rho R^3\left(\frac{dR}{dt}\right)^2 = \frac{4p_0R_0^{\,3}}{3(\gamma - 1)}\left\{1 - \left(\frac{R_0}{R}\right)^{3\gamma-3}\right\},$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{2c_0^{\,2}}{3(\gamma - 1)}\left\{\left(\frac{R_0}{R}\right)^3 - \left(\frac{R_0}{R}\right)^{3\gamma}\right\}, \qquad (71)$$

here

$$c_0 = \sqrt{(p_0/\rho)}. \qquad (72)$$

his quantity c_0, which is of the dimensions of a velocity, is a easure of the rapidity with which the changes take place. It is not ısy to carry the solution further except in the particular case of $= \frac{4}{3}$. If we write

$$R/R_0 = 1 + z, \qquad (73)$$

e have then

$$(1 + z)^2\frac{dz}{dt} = \frac{c_0}{R_0}\sqrt{(2z)}, \qquad (74)$$

hence

$$\frac{c_0t}{R_0} = \sqrt{(2z)}\,(1 + \tfrac{2}{3}z + \tfrac{1}{5}z^2). \qquad (75)$$

his gives the time taken by the radius of the cavity to attain y assigned value R The following table gives a few examples.

R/R_0	c_0t/R_0
1	0
2	2 64
3	6·27
4	11·76
5	19 42

s a concrete illustration, suppose the initial diameter of the cavity be 1 m., and the initial pressure p_0 to be 1000 atmospheres, so at $c_0 = 3{\cdot}16 \times 10^4$ cm /sec. We find that the radius is doubled $\frac{1}{250}$ sec., and multiplied five-fold in about $\frac{1}{30}$ sec. It must be membered that in this investigation, as in the preceding one, the ıter has been assumed to be incompressible With an initial ternal pressure of the order of 10,000 atmospheres, we obtain lues of dR/dt comparable with the velocity of sound in water. he influence of compressibility then ceases to be negligible.

The combination of a source m at a point A and a corresponding sink $-m$ at B gives

$$\phi = \frac{m}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \ldots\ldots\ldots\ldots(76)$$

If we imagine the points A and B to approach one another, whilst the product mBA is constant $(= \mu)$, we have ultimately $r_2 - r_1 =$ AB $\cos\theta$, and

$$\phi = \frac{\mu}{4\pi}\frac{\cos\theta}{r^2} \quad \ldots\ldots\ldots\ldots(77)$$

We have here the conception of a *double-source*. If we combine this with a uniform flow $\phi = -\mathrm{U}x = -\mathrm{U}r\cos\theta$ parallel to Ox, we have

$$\phi = -\left(\mathrm{U}r - \frac{\mu}{4\pi r^2}\right)\cos\theta$$

This makes $-\partial\phi/\partial r = 0$ for $r = a$, provided $\mu = -2\pi\mathrm{U}a^3$. The formula

$$\phi = -\mathrm{U}\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \quad \ldots\ldots(78)$$

therefore gives the steady flow past a sphere of radius a. The tangential velocity at the surface is

$$-\frac{\partial\phi}{r\partial\theta} = -\tfrac{3}{2}\mathrm{U}\sin\theta,$$

and the pressure is accordingly

$$p = \text{constant} - \tfrac{9}{8}\rho\mathrm{U}^2\sin^2\theta. \quad \ldots\ldots\ldots\ldots(79)$$

Since this is the same when θ is replaced by $\pi - \theta$, the resultant effect on the sphere is nil. If we superpose a general velocity $-\mathrm{U}$, we get the case where the sphere is in motion with velocity U in the negative direction of x; thus

$$\phi = -\frac{\mathrm{U}a^3}{2r^2}\cos\theta \quad \ldots\ldots\ldots\ldots(80)$$

If we imagine this motion to be produced instantaneously from rest, the impulsive pressure of the fluid on the sphere, in the direction of x-negative, is

$$\int_0^{\mathrm{R}} \varpi\cos\theta . 2\pi a\sin\theta a d\theta = \int_0^{\pi}\rho\phi\cos\theta\, 2\pi a\sin\theta a d\theta$$
$$= -\tfrac{2}{3}\pi\rho a^3\mathrm{U}, \quad \ldots\ldots\ldots\ldots(81)$$

or $-\frac{1}{2}M'U$, where M′ is the mass of fluid displaced. The impulse which must be given to the sphere to counteract this is $\frac{1}{2}M'U$, and the total impulse in the direction of the velocity is $(M + \frac{1}{2}M')U$, where M is the mass of the sphere itself. It is a proposition in Dynamics that the kinetic energy due to a system of impulses is got by multiplying each constituent of the impulse by the velocity produced in its direction, and taking half the sum of such products. In the present case this gives $\frac{1}{2}(M + \frac{1}{2}M')U^2$. The case is analogous to that of the cylinder, already treated, except that the virtual addition to the mass is $\frac{1}{2}M'$ instead of M′.

This result, viz. that the effect of a *frictionless* liquid on a body moving through it without rotation consists merely in an addition to its inertia, is quite general. Whatever the form of the body, the impulsive pressure necessary to start the actual motion of the fluid instantaneously from rest will evidently be proportional to the velocity (U), and the reaction on the body in the direction of motion will therefore be $-kM'U$, where k is some numerical coefficient. The impulse necessary to be given to the solid is therefore $(M + kM')U$. A similar conclusion would follow from the consideration of the energy produced. The value of k will, of course, depend on the form of the solid and the direction of its motion. The following table gives values for the case of a prolate ellipsoid, the ratio c/a being that of the longer to the shorter semi-diameter. The column under "k_1" relates to motion "end-on", and that under "k_2" to motion "broadside-on".

c/a	k_1	k_2
1 (sphere)	$\frac{1}{2}$	$\frac{1}{2}$
1·5	0·305	0·621
2·0	0·209	0·702
3·0	0·122	0·803
4·0	0·082	0·860
5·0	0·059	0·895
6·0	0·045	0·918
7·0	0·036	0·933
8·0	0·029	0·945
9·0	0·024	0·954
10·0	0·021	0·960
∞ (cylinder)	0	1

Any line AP drawn in a plane through the axis represents an

annular portion of a surface of revolution about Ox. The flux across this portion, say from right to left, will be the same for any two lines from A to P, provided the space between them is occupied by fluid. If A be fixed, this flux will therefore be a function only of the position of P; we denote it by $2\pi\psi$. If PP′ be a linear element δs, drawn in any direction, the flux across the surface generated by its revolution about Ox will be

$$2\pi\delta\psi = q_n . 2\pi y \delta s,$$

where q_n is the velocity normal to δs Hence

$$q_n = \frac{1}{y}\frac{\partial\psi}{\partial s} \ldots\ldots\ldots\ldots\ldots\ldots (82)$$

It was to simplify this formula that the factor 2π was introduced in the definition of ψ. As particular cases of (79), the component velocities parallel and perpendicular to Ox are

$$u = -\frac{1}{y}\frac{\partial\psi}{\partial y}, \quad v = \frac{1}{y}\frac{\partial\psi}{\partial x} \ldots\ldots \qquad (83)$$

The lines for which ψ is constant are stream-lines, and ψ is called the *stream-function*

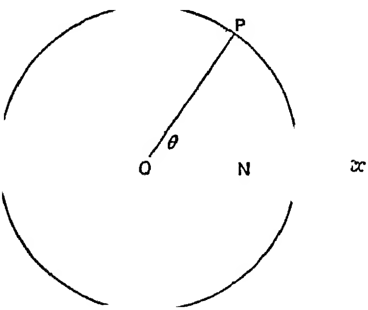


Fig. 16

To find ψ for the case of a point-source, we calculate the flux across the segment of a spherical surface, with OP as radius, cut off by a plane through P perpendicular to Ox The radial velocity across this segment is $m/4\pi r^2$, and the area is $2\pi r(r - x)$, where r = OP, x = ON. Hence, omitting an additive constant, the flux in the desired sense is

$$2\pi\psi = \tfrac{1}{2}mx/r, \text{ or } \psi = \frac{m}{4\pi}\cos\theta \ldots\ldots\ldots \quad (84)$$

The combination of an equal source and sink at A and B gives

$$\psi = \frac{m}{4\pi}(\cos\theta_1 - \cos\theta_2), \ldots\ldots\ldots\ldots\ldots (85)$$

hilst if A and B are made to approach coincidence in such a way hat $m\text{AB} = \mu$, we have ultimately

$$\delta\,(\cos\theta) = -\sin\theta\,\delta\theta = -\sin\theta\,(\text{AB}\sin\theta)/r,$$

and therefore $$\psi = -\frac{\mu}{4\pi}\frac{\sin^2\theta}{r} \quad \ldots\ldots\ldots\ldots (86)$$

or a uniform flow parallel to Ox, we have $2\psi = -Uy^2$, and if e superpose this on (85) or (86) we get stream-line forms, one of hich may be taken as the profile of a stationary solid in the stream. or instance, combining with (86), and putting $\mu = -2\pi Ua^3$,

$$\psi = -\frac{U}{2}\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta \quad \ldots\ldots (87)$$

he line $\psi = 0$ now consists of the circle $r = a$ and of the portions f the axis of x external to it. If we now remove the uniform flow e get the lines of motion due to the sphere moving in the direction f x-negative with velocity U.

The process just indicated admits of great extension By taking series of sources and sinks, not necessarily concentrated in points, long the axis of x, subject to the proviso that the aggregate output zero, and superposing a uniform flow, we may obtain a variety of urves which may serve as the profile of a moving solid This proedure was originated by Rankine from the point of view of naval rchitecture, and has recently been applied to devise profiles which nitate those of airships. Since the motion of the fluid is known, e pressure distribution over the surface can be calculated and ompared with model experiments

7 *Tracing of Stream-lines*

There are various methods by which drawings of systems of tream-lines can be constructed. For example, suppose that the tream-function consists of two parts ψ_1, ψ_2, which are themselves eadily traced Drawing the curves

$$\psi_1 = m\alpha, \quad \psi_2 = n\alpha,$$

here m, n are integers, and α is some convenient constant (the maller the better), these will divide the plane of the drawing into curvilinear) quadrilaterals. The curves

$$\psi_1 \pm \psi_2 = n\alpha$$

will form the diagonals of these quadrilaterals, and are accordingly easily traced if the compartments are small enough. For instance, in the case of (33), where we may put $U = -1$, $a = 1$, without any effect except on the scale of the diagram, we should trace the straight lines

$$r \sin\theta = ma,$$

which are parallel to the axis of x and equidistant, and the circles

$$r = \frac{1}{na} \sin\theta$$

Another method is to write the equation (as above modified) in the form

$$\psi = y\left(1 - \frac{1}{r^2}\right),$$

and to tabulate the function $1/(1 - 1/r^2)$ for a series of equidistant values of r, beginning with unity. This is easily done with the help of Barlow's tables. The values of y where a particular stream-line crosses the corresponding circles are then given by

$$y = \frac{\psi}{1 - \frac{1}{r^2}}.$$

Giving ψ in succession such values as 0·1, 0·2, 0·3, ... a system of stream-lines is readily drawn The same numerical work comes in useful in the case of (39) A similar process can be applied to tracing the stream-lines past a sphere, to which (87) refers

A more difficult example is presented by equation (99) later. Nothing is altered except the scale if we write this in the form

$$\psi = \log\frac{r_1^2}{r_2^2} - x,$$

whence

$$\frac{(x + 1)^2 + y^2}{(x - 1)^2 + y^2} = e^{x+\psi},$$

and therefore

$$r^2 + 1 = 2x\frac{e^{x+\psi} + 1}{e^{x+\psi} - 1} = 2x \coth\frac{x + \psi}{2}.$$

The hyperbolic function on the right-hand has been tabulated, so that we can calculate the values of r (the distance from the origin)

at the points where any given stream-line curve cuts the lines $x =$ constant.

8. *General Equations of Motion*

The general equations of hydrodynamics have so far not been required. To obtain them in their full three-dimensional form we denote by u, v, w the component velocities parallel to rectangular axes at the point (x, y, z) at the time t. They are therefore functions of the four independent variables x, y, z, t. If we fix our attention on a particular instant t_0, their values would gives us a picture of the instantaneous state of motion throughout the field. If on the other hand we fix our attention on a particular point (x_0, y_0, z_0) in the field, their values as functions of t would give us the history of what takes place at that chosen point. We introduce a symbol $\mathrm{D}/\mathrm{D}t$ to denote a differentiation of any property of the fluid considered as belonging to a particular particle. Thus $\mathrm{D}u/\mathrm{D}t$ denotes the component acceleration of a particle parallel to Ox; this is to be distinguished from $\partial u/\partial t$, which is the rate at which u varies at a particular place. The dynamical equations are obtained by equating the rate of change of momentum of a given small portion of the fluid to the forces acting on it. Considering the portion which at time t occupies a rectangular element $\delta x\delta y\delta z$, we have, resolving parallel to Ox

$$\rho\delta x\delta y\delta z\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}\delta x\delta y\delta z - \rho\delta x\delta y\delta z\frac{\partial \mathrm{V}}{\partial x},$$

where the first term on the right hand is the effect of the fluid pressures on the boundary of the element, as determined by (1), whilst the second term is due to extraneous forces (such, for example, as gravity) which are supposed to be conservative, V being the potential energy per unit mass. Thus we find,

$$\left.\begin{aligned} \rho\frac{\mathrm{D}u}{\mathrm{D}t} &= -\frac{\partial p}{\partial x} - \rho\frac{\partial \mathrm{V}}{\partial x}, \\ \rho\frac{\mathrm{D}v}{\mathrm{D}t} &= -\frac{\partial p}{\partial y} - \rho\frac{\partial \mathrm{V}}{\partial y}, \\ \rho\frac{\mathrm{D}w}{\mathrm{D}t} &= -\frac{\partial p}{\partial z} - \rho\frac{\partial \mathrm{V}}{\partial z}. \end{aligned}\right\} \qquad (88)$$

To find expressions for $\mathrm{D}u/\mathrm{D}t$, &c., let P, P′ be the positions occupied by a particle at two successive instants t_1, t_2. Let u_1, u_1'

be the values of u at the points P, P′, respectively, at time t_1, and u_2, u_2' the corresponding values at time t_2. The average acceleration of the particle parallel to Ox in the interval $t_2 - t_1$ is therefore

$$\frac{u_2' - u_1}{t_2 - t_1} = \frac{u_2 - u_1}{t_2 - t_1} + \frac{u_2' - u_2}{t_2 - t_1}.$$

The limiting value of the left-hand side is Du/Dt; that of the first term on the right is $\partial u/\partial t$, the rate of change of u at P. Again, $u_2' - u_2$ is the difference of simultaneous component velocities at the points P′ and P, so that

$$u_2' - u_2 = \frac{\partial u}{\partial s}.\mathrm{PP}' = \frac{\partial u}{\partial s}q(t_2 - t_1),$$

where q is the resultant velocity

$$\sqrt{(u^2 + v^2 + w^2)},$$

and $\partial u/\partial s$ is the gradient of u in the direction PP′. Thus

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + q\frac{\partial u}{\partial s}. \qquad \text{(89)}$$

Now if l, m, n be the direction cosines of δs,

$$\frac{\partial u}{\partial s} = l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z},$$

and $lq = u, mq = v, nw = w$. Hence, finally,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}. \qquad \text{(90)}$$

Similar expressions are obtained for Dv/Dt, Dw/Dt Substituting in (88) we get the dynamical equations in their classical form

To these must be added a kinematical relation, which expresses that the total flux outwards across the boundary of the element $\delta x\delta y\delta z$ is zero. The two faces perpendicular to Ox give $- u\delta y\delta z$, and $(u + \frac{\partial u}{\partial x}\delta x)\delta y\delta z$ respectively, the sum of which is $\partial u/\partial x\delta x\delta y\delta z$ Taking account in like manner of the flux across the remaining faces, and equating the total to zero, we have the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad \text{(91)}$$

of which (7) is a particular case.

When the motion is irrotational we have

$$\frac{\partial u}{\partial y} = -\frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial v}{\partial x},$$

ınd similar relations, so that (90) becomes

$$\frac{Du}{Dt} = -\frac{\partial^2\phi}{\partial x\partial t} + \tfrac{1}{2}\frac{\partial q^2}{\partial x}, \quad \ldots\ldots\ldots\ldots(92)$$

Vhen this is substituted in (88), it is seen that the dynamical equations ıave the integral

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \tfrac{1}{2}q^2 - V + F(t), \quad \ldots\ldots(93)$$

vhere $F(t)$ denotes a function of t only which is to be determined ıy the boundary conditions, but has no effect on the motion. It is vident beforehand that a pressure uniform throughout the liquid, ven if it varies with the time, is without effect The occurrence f $F(t)$ in the present case is merely a consequence of the fact already ıentioned that in an absolutely incompressible fluid changes of ressure are transmitted instantaneously.

The equation of continuity (91) now takes the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \quad \ldots\ldots(94)$$

ı steady motion $\partial\phi/\partial t = 0$, and (93) reduces to our former result ²7).

Vortex Motion

1 *Persistence of Vortices*

Turning now to the consideration of vortex motion, the fundaıental theorem in the subject is that the circulation in any circuit ıoving with the fluid (i e. one which consists always of the same ırticles) does not alter with the time. For, consider any element δx of the integral

$$\int(udx + vdy + wdz),$$

hich expresses the circulation. We have

$$\frac{D}{Dt}(u\delta x) = \frac{Du}{Dt}\delta x + u\frac{D(\delta x)}{Dt} \ldots\ldots\ldots\ldots(95)$$

Now $D(\delta x)/Dt$ is the rate at which the projection on the axis of x, of the line joining two adjacent particles, is increasing, and is therefore equal to δu. Hence,

$$\frac{D}{Dt}(u\delta x) = \frac{Du}{Dt}\delta x + u\delta u = -\frac{1}{\rho}\frac{\partial p}{\partial x}\delta x - \frac{\partial V}{\partial x}\delta x + \tfrac{1}{2}\delta(u^2),$$

and therefore

$$\frac{D}{Dt}(u\delta x + v\delta y + w\delta z) = -\delta\left(\frac{p}{\rho} + V - \tfrac{1}{2}q^2\right). \ldots . (96)$$

When this is integrated round the circuit, the result is zero. Hence

$$\frac{D}{Dt}\int(udx + vdy + wdz) = 0 \ldots \qquad \ldots (97)$$

It is important to notice the restrictions under which this is proved. It is assumed that the density is uniform, that the fluid is frictionless, and that the external forces have a potential. The first of these assumptions is violated, for instance, when convection currents are produced by unequal heating of a mass of water, owing to the variation of density. The second assumption fails when the influence of viscosity becomes sensible.

Irrotational motion is characterized by the property that the circulation is zero in every infinitesimal circuit We now have a general proof that if this holds for a particular portion of fluid at any one instant, it will (under the conditions stated) continue to hold for that particular portion, whether there be rotational motion in other parts of the mass or not. Again, in two-dimensional motion we have seen that the circulation round any small area is equal to the product of the vorticity ζ into the area Since the area occupied by any portion of fluid remains constant as it moves along, we infer that the vorticity also is constant. This has already been proved otherwise in the case of steady motion. The value of ζ is, of course, constant along a line drawn normal to the planes of motion. Such a line is a *vortex-line* according to a general definition to be given presently, and the vortex-lines passing through any small contour enclose what is called a *vortex-filament*, or simply a *vortex*. The *strength* of a vortex is defined by the product of the vorticity into the cross-section, i.e. by the circulation immediately round it.

Still keeping for the moment to the case of two-dimensions, we have seen that the circulation round the boundary of any area

occupied by the fluid is equal to the sum of the circulations round the various elements into which it may be divided, provided these be estimated in a consistent sense. In virtue of the above definitions an equivalent statement is that the circulation in any circuit is equal to the sum of the strengths of all the vortices which it embraces.

2. *Isolated Vortices*

The stream- and velocity-functions due to an isolated rectilinear vortex of strength κ have already been met with in (38) and (58) The velocity distributions due to two or more parallel rectilinear vortices may be superposed. Suppose, for instance, we have a *vortex-pair* composed of two vortices A, B of equal and opposite strengths $\pm\kappa$

Fig 17

Each produces in the other a velocity $\kappa/2\pi a$, where a is the distance apart, at right angles to AB. The pair advances therefore with this constant velocity, the distance apart being unaltered. The lines of flow are given by

$$\psi = \frac{\kappa}{2\pi}\log\frac{r_1}{r_2}, \qquad \ldots\ldots \quad (98)$$

where r_1, r_2 are the distances of A, B respectively from the point P to which ψ refers The lines for which the ratio r_1/r_2 has the same value are co-axial circles having A, B as limiting points If we superpose a uniform flow $\kappa/2\pi a$ in the direction of y negative, the case is reduced to one of steady motion, and the stream-function is now

$$\psi = \frac{\kappa}{2\pi}\left(\log\frac{r_1}{r_2} - \frac{x}{a}\right). \qquad (99)$$

The stream-line $\psi = 0$ consists partly of the axis of y, where $r_1 = r_2$ and $x = 0$, and partly of a closed curve which surrounds always the same mass of the fluid. This portion is carried forward by the vortex-pair in the original form of the problem.

If a flat blade, e g a paper-knife, held vertically, be dipped into water, and moved at right angles to its breadth for a short distance, and then rapidly withdrawn, a vortex-pair will be produced by friction at the edges, and will be seen to advance in accordance with the preceding theory. The positions of the vortices are marked by

the dimples produced on the water surface In this way the action of one vortex-pair on another may be studied.

The detailed study of vortex motion in three dimensions would lead us too far, but a brief sketch of the fundamental relations may be given. It is necessary in the first place to introduce the notion of vorticity as a vector. Through any point P we draw three lines PA, PB, PC parallel to the co-ordinate axes, meeting any plane drawn infinitely near to P in the points A, B, C. It is evident at once from the figure that the circulation round ABC is equal to the sum of the circulation round the triangles PBC, PCA, PAB, provided the positive direction of the circulations be right-handed as regards the positive directions of the co-ordinate axes. Now, if

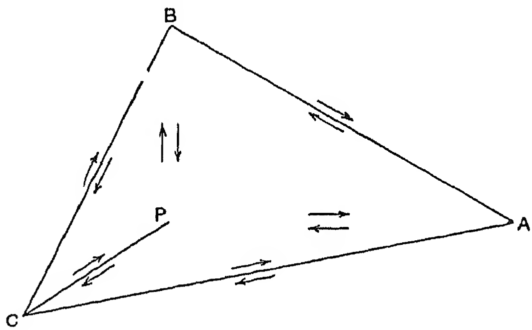


Fig 18

l, m, n be the direction-cosines of the normal drawn from P to the plane ABC, and Δ the area ABC, the areas of the above triangles are $l\Delta$, $m\Delta$, $n\Delta$, respectively. Hence if ξ, η, ζ be the vorticities in these planes, i.e. the ratios of these circulations to the respective areas, the circulation round ABC will be

$$(l\xi + m\eta + n\zeta)\Delta \qquad \ldots(100)$$

We may regard ξ, η, ζ as the components of a vector ω, and the expression (100) is then equal to $\omega \cos\theta \, \Delta$ where θ is the angle which the normal to Δ makes with the direction of ω. In other words, the vorticity in any plane is equal to the component of ω along the normal to that plane.

The value of ζ has been given in (12). Writing down the corresponding formulæ for ξ and η, we have altogether

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \ldots(101)$$

We have, of course,

$$\omega^2 = \xi^2 + \eta^2 + \zeta^2 \dots\dots\dots\dots\dots (102)$$

A line drawn from point to point always in the direction of the vector (ξ, η, ζ) is called a *vortex-line*. The vortex-lines which meet any given curve generate a surface such that the circulation in every circuit drawn on it is zero. If the curve in question be closed, and infinitely small, the fluid enclosed by the surface constitutes a *vortex-filament*, or simply a *vortex*. Consider a circuit such as ABCAA′C′B′A′A in the figure, drawn on the wall of the filament. Since the circulation in it is zero, and since the portions due to AA′ and A′A cancel, the circulation round ABC is equal to that round A′B′C′. Supposing the planes of these two curves to be cross-sections of the filament, we learn that the product of the resultant vorticity into the cross-section has the same value along the vortex. This product is called the *strength* of the vortex. The dynamical theorem above proved shows that under the conditions postulated the strength of a vortex does not vary with the time. The constancy of the strength of a *vortex-ring* has already been proved in the case of steady motion.

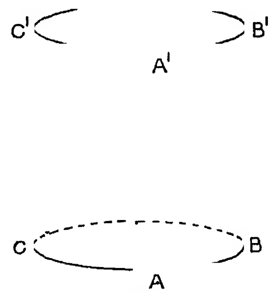


Fig 19

The argument by which the circulation in a plane circuit was, under a certain condition, proved to be equal to the sum of the strengths of all the vortices which it embraces, is easily extended (under a similar condition) to the general case.

The most familiar instance of isolated vortices is that of smoke rings, which are generated in the first instance by viscosity, but retain a certain degree of persistence. A vortex-ring at a distance from other vortices, or from the boundaries of the fluid, advances along its axis with uniform velocity. The mutual influence of vortex-rings is closely analogous to that of vortex-pairs.

Wave Motion

1. *Canal Waves*

Water-waves are by no means the simplest type of wave-motion met with in Mechanics, and the general theory is necessarily somewhat intricate, even when we restrict ourselves to oscillations of

small amplitude. The only exception is in the case of what are variously called *long waves*, or *tidal waves*, or *canal waves*, the characteristic feature being that the wave-length is long compared with the depth, and the velocity of the fluid particles therefore sensibly uniform from top to bottom.

Taking this case first, we inquire under what condition a wave can travel without change of form, and therefore with a definite velocity. Supposing this velocity to be c, from left to right, we may superpose a general velocity $-c$ in the opposite direction and so reduce the problem to one of steady motion. The theory is now the same as for the flow through a pipe of gradually varying section, except that the upper boundary is now a free surface, instead of a rigid wall. If h be the original depth, the velocity where the surface-elevation is η will be

$$q = -\frac{ch}{h+\eta}. \qquad \text{(103)}$$

The pressure along the wave-profile, which is now a stream-line, is given by Bernoulli's equation

$$\frac{p}{\rho} = \text{constant} - \tfrac{1}{2}q^2 - g\eta = \text{constant} - \tfrac{1}{2}c^2\left(1 + \frac{\eta}{h}\right)^{-2} - g\eta$$

$$= \text{constant} - \tfrac{1}{2}c^2\left(1 - \frac{2\eta}{h}\right) - g\eta, \qquad \text{(104)}$$

approximately, if we neglect the square of η/h. This pressure will be independent of η, provided

$$c = \sqrt{(gh)}. \qquad \text{(105)}$$

The required wave-velocity is therefore that which would be acquired by a particle falling vertically under gravity, from rest, through a space equal to half the depth.

If we now restore the original form of the problem, by imposing a velocity c in the positive direction, we have

$$q = c - \frac{ch}{h+\eta} = c\frac{\eta}{h}, \qquad \text{(106)}$$

approximately. The velocity of the water itself is therefore forward or backward, according as η is positive or negative, i.e. it is forward where there is an elevation, and backward where there is a depression. The potential energy per unit area of the surface is $\frac{1}{2}g\rho\eta^2$, and the corresponding kinetic energy is $\frac{1}{2}\rho q^2 h = \frac{1}{2}\rho c^2\eta^2/h$. Since these are

qual by (106), the energy of a progressive wave is half-potential nd half-kinetic.

The condition for permanence of form has not, of course, been xactly fulfilled in the above calculation. A closer approximation o fact is evidently obtained if in (105) we replace h by $h + \eta$; this vill give us the velocity of the wave-form relative to the water in he neighbourhood, which is itself moving with the velocity given y (106), if η/h is small. The elevation η is therefore propagated *n space* with the velocity

$$\sqrt{\{g(h+\eta)\}} + \sqrt{(gh)}\frac{\eta}{h} = \sqrt{(gh)}\left(1 + \frac{3}{2}\frac{\eta}{h}\right), \qquad \ldots\ldots(107)$$

ipproximately. The more elevated portions therefore move the faster, with the result that the profile of an elevation tends to become steeper in front and more gradual in slope behind.

2. *Deep-water Waves*

Proceeding to the more general case, we will assume that the motion takes place in a series of parallel vertical planes, and is the same in each of these, so that the ridges and furrows are rectilinear. Fixing our attention on one of these planes, we take rectangular axes Ox, Oy, the former being horizontal, and the later vertical with the positive direction upwards. The problem being reduced to one of steady motion as before, the stream-function will be

$$\psi = cy + \psi_1, \qquad .(108)$$

where ψ_1 is supposed to be small By Bernoulli's equation

$$\frac{p}{\rho} = \text{constant} - gy - \frac{1}{2}\left\{\left(c + \frac{\partial\psi_1}{\partial y}\right)^2 + \left(\frac{\partial\psi_1}{\partial x}\right)^2\right\}$$
$$= \text{constant} - gy - c\frac{\partial\psi_1}{\partial y}, \qquad \ldots(109)$$

if we neglect small terms of the second order. We assume the motion to have been originated somehow by the operation of ordinary forces, and therefore to be irrotational, so that

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = 0. \qquad \ldots(110)$$

We further assume, in the first instance, that the depth is very great

compared with the other linear magnitudes with which we are concerned. The simplest solution of (110) which is periodic with respect to x, and vanishes for $y = -\infty$, is

$$\psi_1 = Ce^{ky} \sin kx. \quad\quad (111)$$

If we take the origin O at the mean level of the surface, the condition that the wave-profile may be a stream-line is, by (108),

$$\eta = -\frac{C}{c}e^{ky} \sin kx = -\frac{C}{c} \sin kx, \quad\quad (112)$$

if we neglect an error of the second order in C. We have still to secure that this is a line of constant pressure. Substituting in (109), the result will be independent of x, provided

$$g\frac{C}{c} - kcC = 0, \text{ or } c^2 = \frac{g}{k}, \quad\quad (113)$$

to our order of approximation. The wave-length, i.e. the distance between successive crests or hollows is $\lambda = 2\pi/k$, so that

$$c = \sqrt{\left(\frac{g\lambda}{2\pi}\right)}. \quad\quad (114)$$

This gives the wave-velocity relative to still water

The original form of the problem is restored if we omit the first term in (108), and replace x by $x - ct$. Thus, if we denote the surface amplitude C/c by a we have

$$\left.\begin{aligned} \eta &= a \sin k(ct - x), \\ \psi &= -ace^{ky} \sin k(ct - x). \end{aligned}\right\} \quad\quad (115)$$

To find the motion of the individual particles, we may with consistent approximation write

$$\left.\begin{aligned} \frac{Dx}{Dt} &= u = -\frac{\partial\psi}{\partial y} = kac\, e^{ky_0} \sin k(ct - x)_0, \\ \frac{Dy}{Dt} &= v = \frac{\partial\psi}{\partial x} = kac\, e^{ky_0} \cos k(ct - x_0), \end{aligned}\right\} \quad\quad (116)$$

where (x_0, y_0) is the mean position of the particle referred to. Integrating with respect to t, and recalling (113), we have

$$\left.\begin{aligned} x &= x_0 - a\, e^{ky_0} \cos k(ct - x), \\ y &= y_0 + a\, e^{ky_0} \sin k(ct - x). \end{aligned}\right\} \quad\quad (117)$$

ıe particles therefore describe circles whose radius $a\, e^{ky_0}$ diminishes m the surface downwards. At a depth of a wave-length, $y_0 = -\lambda$, $= e^{-2\pi} = 0{\cdot}00187$. The preceding investigation is therefore ıctically valid for depths of the order of λ, or even less.

For smaller depths, provided they are uniform, the solution (111) to be replaced by

$$\psi_1 = \mathrm{C} \sinh k(y + h) \sin kx, \quad \ldots\ldots\ldots (118)$$

ıce this makes $v = 0$ for $y = -h$. We should now find

$$c^2 = \frac{g}{k} \tanh kh = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda} \quad \ldots\ldots (119)$$

ır small values of h/λ this gives $c = \sqrt{(gh)}$, and so verifies the ·mer theory of long waves As h/λ increases, $\tanh kh$ tends to ıity as a limit, and we reproduce the result (114). The paths of the dividual particles are ellipses whose semi-axes

$$a \frac{\cosh k(y_0 + h)}{\sinh kh}, \quad a \frac{\sinh k(y_0 + h)}{\sinh kh}$$

e horizontal and vertical, respectively

The energy, per unit area of the surface, of deep-water waves is und as follows The potential energy is

$$\tfrac{1}{2} g\rho\eta^2 = \tfrac{1}{2} g\rho a^2 \sin^2 k(ct - x), \quad . \quad (120)$$

e mean value of which is $\frac{1}{4} g\rho a^2$ The kinetic energy is

$$\int_{-\infty}^{0} \tfrac{1}{2}\rho(u^2 + v^2) dy = \tfrac{1}{2}\rho k^2 a^2 c^2 \int_{-\infty}^{0} e^{2ky} dy = \tfrac{1}{4}\rho k a^2 c^2 \ldots (121)$$

r (115) Since $c^2 = g/k$ the energy is, on the whole, half-potential ıd half-kinetic The total energy per wave-length $(2\pi/k)$ is $\pi\rho a^2 c^2$. his is equal to the work which would be required to raise a stratum the fluid, of thickness a, through a height $\frac{1}{2}a$.

The theory of waves on the common boundary of two superosed liquids, both of great depth, is treated in a similar manner. he formulæ (108), (109), (111) may be retained as applicable to ıe lower fluid. For the upper fluid (of density ρ') we write

$$\psi = cy + \psi_1', \quad \ldots\ldots\ldots (122)$$

ıd

$$\psi_1' = \mathrm{C}' e^{-ky} \sin kx \ldots \quad \ldots\ldots\ldots (123)$$

since ψ_1' must vanish when y is very great. This makes

$$\eta = -\frac{C'}{c}\sin kx, \quad \ldots\ldots\ldots\ldots\ldots (124)$$

We have also

$$\frac{p}{\rho'} = \text{constant} - c\frac{\partial\psi_1'}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots (125)$$

The two values of p will be equal provided

$$g\rho\frac{C}{c} - kc\rho C = g\rho\frac{C'}{c} + kc\rho' C' \quad \ldots\ldots\ldots (126)$$

By comparison of (112) and (124) we have $C = C'$, and therefore

$$c^2 = \frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} \quad \ldots\ldots\ldots\ldots\ldots (127)$$

If $(\rho - \rho')/(\rho + \rho')$ is small, as in the case of oil over water, the oscillations are comparatively slow, owing to the relative smallness of the potential energy involved in a given deformation of the common surface. A remarkable case in point is where there is a stratum of fresh water over salt, as in some of the Norwegian fiords, where an exceptional wave-resistance due to this cause is sometimes experienced

The preceding theory of surface-waves is restricted to the case of a simple-harmonic profile It is true that any other form can be resolved into simple-harmonic constituents of different wave-lengths, and that it is legitimate, so far as our approximation extends, to superpose the results. But the formula (114) shows that each constituent will travel with its own velocity, so that the form of the profile continually changes as it advances. The only exception is when the wave-lengths which are present with sensible amplitude are all large compared with the depth, in which case there is a common wave-velocity $\sqrt{(gh)}$ as found above.

3. *Group Velocity*

One consequence of the dependence of wave-velocity on wave-length is that a group of waves of approximately simple-harmonic type often appears to advance with a velocity less than that of the individual waves. The simplest illustration is furnished by the

combination of two simple-harmonic trains of equal amplitude but slightly different wave-lengths, thus

$$\eta = a\cos k(x - ct) + a\cos k'(x - c't)$$
$$= 2a\cos\left(\frac{k-k'}{2}x - \frac{kc-k'c'}{2}t\right)\cos\left(\frac{k+k'}{2}x - \frac{kc+k'c'}{2}t\right)..(128)$$

If k and k' are nearly equal, the first trigonometrical factor oscillates very slowly between $+ 1$ and $- 1$ as x is varied, whilst the second factor represents waves travelling with velocity $(kc + k'c')/(k + k')$. The surface has therefore the appearance of a series of groups of waves separated by bands of nearly smooth water. It is evident then that the motion of each group will be practically independent of the rest. The centre of one of the groups is determined by

$$\frac{k-k'}{2}x - \frac{kc-k'c'}{2}t = 0;$$

the group as a whole is therefore propagated with the velocity

$$U = \frac{kc-k'c'}{k-k'} = \frac{d(kc)}{dk}, \quad \ldots\ldots\ldots \quad (129)$$

in the limit This is called the *group-velocity* If c is constant, as when the wave-length is large compared with the depth, we have $U = c$. On the other hand, for waves on deep water, $c^2 = g/k$, by (113), so that

$$\frac{2}{c}\frac{dc}{dk} = -\frac{1}{k},$$

whence $$U = \tfrac{1}{2}c, \quad \ldots\ldots (130)$$

or the group-velocity is only one-half the wave-velocity The general formula, obtained from (119), is

$$\frac{U}{c} = \tfrac{1}{2} + \frac{kh}{\sinh 2kh} \quad . \ (131)$$

This expression diminishes from 1 to $\frac{1}{2}$ as kh increases from 0 to ∞.

The group-velocity U determines the rate of propagation of energy across a vertical plane. To take the case of deep-water waves as simplest, the rate at which work is done on the fluid to the right

of a plane through the origin perpendicular to the axis of x is

$$\int_{-\infty}^{0} pudy \quad \ldots \qquad \ldots\ldots \qquad \ldots(132)$$

The value of p is given by Bernoulli's equation provided we put $q^2 = (-c + u)^2 + v^2 = c^2 - 2cu$, to our order of approximation. The only term in the resulting value of p which varies with the time is ρcu. Now

$$\rho c\int_{-\infty}^{0} u^2dy = \rho k^2a^2c^3 \sin^2 kct\int_{-\infty}^{0} e^{2ky}dy = \tfrac{1}{2}\rho ka^2c^3 \sin^2 kct \ldots(133)$$

The work done in a complete period $(2\pi/kc)$ is therefore $\frac{1}{2}\pi\rho a^2c^2$, which is *half* the energy of the waves which pass the above plane in the same time. The apparent paradox disappears if we remember that the conception of an infinitely extended train is an artificial one. In the case of a finite train, generated by some periodic action at the origin which has only been in operation for a finite time, the profile will cease to be approximately uniform in character and sinusoidal near the front, there will be a gradual diminution of amplitude, and increase of wave-length, by which the transition to smooth water is effected. We infer from the preceding argument that the approximately simple-harmonic portion of the train is lengthened only by half a wave-length in each period of the originating force.

The principle that U rather than c determines the rate of propagation of energy holds also, not only in the case of waves on water of finite depth, but in all cases of wave-motion in Physics.

Some further results of theory must be merely stated in general terms. A localized disturbance travelling over still water with velocity c leaves behind it a train of waves whose length $(2\pi/k)$ is related to c by the formula (113) or (119), as the case may be. In the same way a stationary disturbance in a stream produces a train of waves on the down-stream side. In the former case the energy spent in producing the train measures the *wave-resistance* experienced by the disturbing agency. If E be the mean energy per unit length of the wave-train, the space in front of the disturbance gains in unit time the energy cE, whilst the energy transmitted is UE, where U is the group-velocity. The wave-resistance R is therefore given by

$$Rc = (c - U)E \ldots\ldots\ldots\ldots\ldots(134)$$

'he value of E has been found to be $\frac{1}{2}g\rho a^2$, but unfortunately the alue of a can be predicted only in a few rather artificial cases.

A curious point arises in the case of finite depths. It appears ·om (119) that the wave-velocity cannot exceed $\sqrt{(gh)}$. The above tatements do not apply, therefore, if the speed of the travelling isturbance exceeds this limit The effect is then purely local, and $\mathrm{E} = 0$. A considerable diminution in resistance was in fact observed y Scott Russell when the speed of a canal boat was increased in ιis way; and an analogous phenomenon has been noticed in the ιse of torpedo boats moving in shallow water.

Viscosity

1. *General Equations*

The subject of viscosity is treated in Chapter III, which deals ιainly with cases of steady motion where this influence is preominant The general equations of motion of a viscous fluid ave the forms

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} - \rho\frac{\partial \mathrm{V}}{\partial x} + \mu\nabla^2 u, \quad \ldots\ldots \quad (135)$$

ιth two similar equations in (v, y) and (w, z), where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2.$$

he formal proof must be passed over, but an interpretation of the ιuations, which differ only from (88) by the terms at the ends, n be given as follows. Considering any function of the position a point, let F be its value at P, whose co-ordinates are (x, y, z). s value at an adjacent point $(x + \alpha, y + \beta, z + \gamma)$ will exceed its ılue at P by the amount

$$\frac{\partial \mathrm{F}}{\partial x}\alpha + \frac{\partial \mathrm{F}}{\partial y}\beta + \frac{\partial \mathrm{F}}{\partial z}\gamma$$
$$+ \frac{1}{2}\left(\frac{\partial^2\mathrm{F}}{\partial x^2}\alpha^2 + \frac{\partial^2\mathrm{F}}{\partial y^2}\beta^2 + \frac{\partial^2\mathrm{F}}{\partial z^2}\gamma^2 + 2\frac{\partial^2\mathrm{F}}{\partial y\partial z}\beta\gamma + 2\frac{\partial^2\mathrm{F}}{\partial z\partial x}\gamma\alpha + 2\frac{\partial^2\mathrm{F}}{\partial x\partial y}\alpha\beta\right),$$

proximately. If we integrate this over the volume of a sphere small radius r having P as centre, the first three terms give a ro result owing to the cancelling of positive and negative values α, β, γ. The terms containing $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, also disappear for a nilar reason. The mean value of α^2 or β^2 or γ^2 on the other hand

is $\frac{1}{5}r^2$, by the theory of moments of inertia. The mean value over the sphere of the aforesaid excess is therefore $\frac{1}{10}r^2\nabla^2F$. The reason why this should vary with the radius of the sphere is obvious. It is also clear that the expression ∇^2F gives a measure of the degree to which the value of the function F in the immediate neighbourhood of P deviates from its value at P. In particular ∇^2u measures the extent to which the x-component of the velocity in the neighbourhood of P exceeds the component at P. The first of the equations (135) accordingly asserts that in addition to the forces previously investigated there is a force proportional to this measure. An excess of velocity about P contributes a force tending to drag the matter at P in the direction of this excess.

The coefficient μ in (135) is called the *coefficient of viscosity*. In cases of varying motion we are often concerned not so much by the viscosity itself as by the ratio which it bears to the inertia of the fluid. It is then convenient to introduce a symbol (ν) for the ratio μ/ρ This is called the *kinematic viscosity*.

An important conclusion bearing on the comparison of model- and full-scale experiments can be drawn from the mere *form* of these equations Omitting the term representing extraneous force, the first equation is in full

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \quad \ldots(136)$$

Now consider another state of motion which is exactly similar except for the altered scales of space and time. Distinguishing this by accented letters, a comparison of corresponding terms in the respective equations shows that we must have

$$\frac{u'}{t'}:\frac{u}{t} = \frac{u'^2}{x'}:\frac{u^2}{x} = \frac{p'}{\rho' x'}:\frac{p}{\rho x} = \frac{\nu' u'}{x'^2}\cdot\frac{\nu u}{x^2}\ldots\ldots(137)$$

The equality of the first two ratios requires that

$$u':u = \frac{x'}{t'}:\frac{x}{t},$$

as was evident beforehand. The equality of the second and fourth ratios requires

$$\frac{\nu'}{u'x'} = \frac{\nu}{ux}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(138)$$

A necessary condition for the similarity of the two motions is therefore that Vl/ν should have the same value in both, where V is any

racteristic velocity, and l any linear dimension involved. The o of corresponding stresses is then

$$\frac{p'}{p} = \frac{\rho' u'^2}{\rho u^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (139)$$

It is to be noted that the viscous terms disappear from the ations (135) if the motion is irrotational, since we then have $= 0$, and therefore $\nabla^2 u = 0$, $\nabla^2 v = 0$, $\nabla^2 w = 0$. But it is eneral impossible to reconcile the existence of irrotational motion 1 the condition of no slipping at the boundary, which is well blished experimentally. The above remark suggests, however, , when the motion is started, vorticity originates at the boundary is only gradually diffused into the interior of the fluid.

2. *Two-dimensional Cases*

The diffusion of vorticity is most easily followed in the two-ensional case The equations may be written, in virtue of , in the forms

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - v\zeta &= -\frac{\partial \chi}{\partial x} + \nu^2\nabla_1 u, \\ \frac{\partial v}{\partial t} + u\zeta &= -\frac{\partial \chi}{\partial y} + \nu\nabla_1^2 v, \end{aligned}\right\} \ldots \quad (140)$$

where
$$\chi = \frac{p}{\rho} + \tfrac{1}{2}(u^2 + v^2) + V, \ldots\ldots\ldots \quad (141)$$

and
$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad \ldots \quad (142)$$

ientiating the second of equation (140) with respect to x, and irst with respect to y, and subtracting and making use of the tion of continuity (7), we have, finally

$$\frac{D\zeta}{Dt} = \nu\nabla_1^2\zeta. \ldots\ldots\ldots\ldots (143)$$

is exactly the equation of conduction of heat, with the vorticity place of the temperature, and the *kinematic viscosity* $\nu (= \mu/\rho)$ ace of the *thermometric conductivity*. Consequently, various n results in the theory of conduction can be at once utilized e present connection.

For instance, the known solution for the diffusion of heat from an initially heated straight wire into a surrounding medium can be applied to trace the gradual decay of a line vortex initially concentrated in the axis of z. Since there is symmetry about Oz the equation (143) takes the form

$$\frac{\partial \zeta}{\partial t} = \nu\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right), \quad \ldots \ldots (144)$$

as may be seen by a comparison of the left-hand members of (25) and (26) It is easily verified by differentiation that this equation is satisfied by

$$\zeta = \frac{\kappa}{4\pi\nu t}e^{-r^2/4\nu t}, \quad \ldots \ldots (145)$$

which vanishes for $t = 0$ except at the origin. Moreover, this gives for the circulation in a circle of radius r

$$\int_0^r \zeta \, 2\pi r dr = \kappa(1 - e^{-r^2/4\nu t}) \quad \ldots \quad (146)$$

As t increases from 0 to ∞, this sinks from κ to 0. The value of ζ, on the other hand, at any given distance r increases from zero to a maximum and then falls asymptotically to zero.

A comparatively simple application of the equations of motion is to the case of "laminar" flow in parallel planes, or of smooth rectilinear flow in pipes, but the results have only a restricted application to actual phenomena To take an example due to Helmholtz, consider the flow of a hypothetical atmosphere of uniform density, and height H, over a horizontal plane If it is subject to inertia and viscosity alone, the equation of motion is

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \quad \ldots \quad (147)$$

with the conditions that $u = 0$ for $y = 0$ and $\partial u/\partial y = 0$ for $y = \mathrm{H}$. These are all satisfied by

$$u = \mathrm{A}e^{-\nu k^2 t}\sin ky \quad \ldots \ldots (148)$$

provided

$$\cos k\mathrm{H} = 0, \quad \text{or} \quad k = (2n + 1)\frac{\pi}{2\mathrm{H}}, \quad (149)$$

where n is an integer. By addition of such solutions with different values of n and suitable values of the coefficients A we can represent

the effect of any initial state, e.g. one of uniform velocity. The most persistent constituent in the result is that for which $n = 0$. This will have fallen to one-half its original value when

$$\nu k^2 t = \log 2, \quad \text{or} \quad t = \frac{4 \log 2}{\pi^2} \frac{H^2}{\nu} \qquad \text{(150)}$$

Putting $\nu = 0{\cdot}134$ (air), $H = 8026$ metres, this makes $t = 305{,}000$ years! The fact is that in such a case the laminar motion would be unstable, turbulent motion would ensue, by which fresh masses of fluid moving with considerable velocity are continually brought into contact with the boundary, so that the influence of viscosity is enormously increased.

CHAPTER III

Viscosity and Lubrication

A. VISCOSITY

All motions of actual fluids, as distinguished from the "perfect fluid" of the mathematician, are accompanied by internal forces which resist the relative movements and are therefore analogous to frictional forces between solid bodies The origin of the frictional resistances is in all cases referred to the property of *viscosity*, common in varying degree to all fluids, which has already been defined in general terms in Chapter I. The present chapter is devoted to a fuller explanation of the theory of this property and to discussions of some of its direct applications, one of the chief of these being to the theory of lubrication

There are other direct applications of the theory of viscosity which are of importance to engineers, most though not all of which relate to the motion of fluids in narrow channels or in thin layers between solid surfaces, and these applications are met with in all branches of engineering. The fluid frictions, however, which chiefly concern hydraulic and other engineers, who deal with fluids such as water or air in large volumes, though physically referable in origin to viscosity, cannot be directly calculated by means of its theory The appropriate methods applicable to such cases are discussed in Chapters IV and V. In the meantime it may be said of the *direct* applications of the theory, in Rayleigh's words (20, p 159),* that in these cases "we may anticipate that our calculations will correspond pretty closely to what actually happens—more than can be said of some branches of hydrodynamics".

* Arabic numerals in brackets after names of authors refer to the short bibliography at the end of this chapter.

Laminar Motion

The law of viscous resistance is most clearly conceived in the se of *laminar motion*, which may be defined as a state of motion of body of fluid in which the direction of the motion of the particles the same at all points and the velocity is the same throughout ch of a series of planes parallel to one another and to the direction motion A volume of fluid in laminar motion can thus be roughly garded as a series of very thin yers of solid material, sliding one ɔon another in a common direc- ɔn Quantitatively, if the face ', δy of the rectangular element ', δy, δz (fig 1) is parallel to the ninæ, and if the laminar motion in the direction of X, the velo- y of flow, u, at any point P, ll depend only on the distance, of the point P from the plane Y If the element is sufficiently nall, u may be taken as varying uniformly with z over the small stance δz, so that if u_0, u_1 are respectively the velocities of the ninæ which form the lower and upper faces of the element $= u_0 + \frac{\partial u}{\partial z}\delta z$, in which $\frac{\partial u}{\partial z}$ can be regarded as constant over e small distance δz

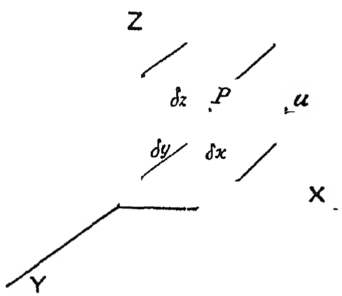


Fig 1

In a viscous fluid there will then be exerted a shearing force, or iction, parallel to X, between the portions of the element of fluid ove and below any section of the element parallel to the face δy, tending to retard the portion which is moving with the higher locity, and the magnitude of this force will be

$$S = \mu\frac{\partial u}{\partial z}\delta x\delta y, \ldots \qquad \ldots\ldots \qquad (1)$$

being a quantity, independent of x, y, z, u, and $\frac{\partial u}{\partial z}$, known as the *efficient of viscosity*.

Coefficient of Viscosity

The value of the quantity μ varies greatly from one fluid to another, and in any one fluid it changes with the temperature, and to a smaller extent with the pressure, of the fluid. Its value is in general much higher for liquids than for gases Liquids in which the value of μ is low are said to be "limpid", "thin", or "light", while those in which it is comparatively great are said to be "viscous", "thick" or "heavy". There is however no necessary, or general, correspondence between the density of a liquid and its viscosity. Thus mercury, the heaviest of known liquids at atmospheric temperatures, is one of the least viscous.

The fact that the coefficient of viscosity, for a given liquid at constant temperature, is independent of the rate of shear was first experimentally proved with great accuracy by Poiseuille (1), not, however, by direct measurement of plane laminar flow, but by investigation of the flow of water in small cylindrical tubes

The flow of fluids in such tubes, as well as the motion of viscous fluids in many other cases which are of practical interest, is closely analogous to plane laminar flow.

The importance of the coefficient of viscosity μ, however, arises from the fact that it is the sole physical constant connecting the internal frictional resistances of fluids with their relative motions, not only in the case of such simple types of motion, but of all kinds of fluid motion however complicated, provided that they are not discontinuous or unstable

The explanation of this unique property of the coefficient of viscosity requires some analysis of the types of deformation of which a fluid element is susceptible This analysis is given briefly in the following paragraphs, from which it will be seen that the relations between the internal motions and stresses in a fluid are similar to, but essentially simpler than, those between the deformations and stresses in an elastic body The failure, already referred to, of the law of viscosity when fluid motions become discontinuous or unstable may be regarded as analogous to the failure of the laws of elasticity in solids when fracture takes place or the "yield-point" is exceeded. In such cases the conditions which result are no longer amenable to theoretical calculation.

We proceed to show that, when no such discontinuities exist, there is in fluids only one kind of internal resistance and only one coefficient of viscosity.

Relative Velocities

If u, v, w (see fig 2) are the components of the velocity parallel to the rectangular axes X, Y, Z of a particle of the fluid at the point x, y, z, the corresponding components for the neighbouring point $x + \delta x, y + \delta y, z + \delta z$ are

$$\left.\begin{aligned} u' &= u + \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z, \\ v' &= v + \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial z}\delta z, \\ w' &= w + \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z; \end{aligned}\right\} \quad (2)$$

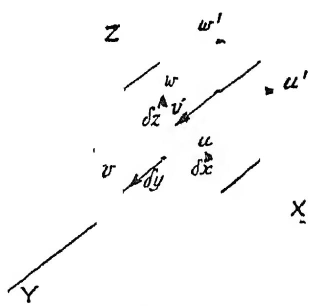


Fig 2

and the components of the velocity of the second point relatively to the first are $u' - u$, $v' - v$, $w' - w$, or

$$\left.\begin{aligned} &\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z, \\ &\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial z}\delta z, \\ &\frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Of the derivatives in these expressions it is clear from inspection of fig 2 that $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$, and $\frac{\partial w}{\partial z}$ represent rates of stretching or elongation of the element in the directions of X, Y, and Z respectively, while by the pairs of sums of derivatives:

$$\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

are represented respectively rates of change of the angles between the edges δy and δz, δz and δx, and δx and δy of the element

Thus by means of these six expressions any deformation of the element can be expressed.

As a hypothesis which is suggested as probable by the experimental law proved by Poiseuille, but which depends for its real justification on the consistent correspondence of the results of theory with experience, it is assumed that the frictional forces arise from

he rates of deformation of the elements of the fluid, and are linear unctions of these rates.

As to the three rates of elongation $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$, it is a well-known heorem that they can be resolved into a rate of dilatation or com- pression of the elementary volume, uniform in all three directions, combined with three rates of shearing deformation respectively in he directions of the diagonals of the faces of the element supposed ubical.*

As there is no experimental evidence of any internal resistances, either in liquids or gases, depending on rates of change of volume by dilatations or compressions equal in all directions, resistance to n elongation, such as $\frac{\partial u}{\partial x}\delta x$, can only arise from its shearing com- ponents. Such resistances are therefore of the same kind as those which depend on the purely shearing deformations whose rates are $\frac{w}{y} + \frac{\partial v}{\partial z}$, &c

In the applications which follow, the axes X, Y, Z will be so hosen that the rates of elongation, such as $\frac{\partial u}{\partial x}$, and consequently also heir component rates of shear, are everywhere small compared to he rates of shear represented by $\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$, &c.

Now in a homogeneous liquid or gas there is no physical difference n the properties depending on the direction of the co-ordinates, onsequently (the frictional forces being linear functions of the rates f shear) the only forces that will arise may be expressed as:

$$\left.\begin{aligned} {}^{\dagger}S_{yz} &= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\ S_{zx} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ S_{xy} &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (4)$$

nvolving the single coefficient μ By comparison with fig. 1, and

* Cf the similar theorem for stresses (Morley, *Strength of Materials*, 2nd ed, p 12).

† In this notation the first subscript indicates the direction of the normal to the ane on which the force acts, the second the direction in which the force acts— us S_{yz} is the shearing stress on a plane a normal of which is parallel to Oy and z acts in the Oz direction

equation (1) with the second of the above equations (4), in ich $\frac{\partial w}{\partial x}$ is taken as zero, it is seen that this constant is the same the coefficient μ introduced in the special case of laminar ·tion.

Conditions at the Bounding Surfaces of Fluids

Before the laws of fluid friction can be applied to fluids as we ually have to deal with them, account must be taken of the be- riour of the fluid where it is in contact with the solid bodies which itain it. In the case of liquids the condition of a free upper face, usually a surface of contact with air at atmospheric pressure, also to be considered.

It is clear in the first place that the presence of a boundary in- ves that on the bounding surface the itive velocity of the fluid normal to t surface is zero. The normal velo- ' will furthermore be very small at points *near* the bounding surface. , let W, fig 3, be the fixed bound- surface, and W′ a surface in the d parallel, and very near, to W. simplicity W and W′ may be sidered plane. Let the average velocity towards W over a le of radius R in the plane W′ be v, the normal distance between and W′ being δn. Then a volume of fluid $\pi R^2 v$ flows through circular area in unit time In the same time a volume $2\pi R\delta n.\sigma$ rs outward between the surfaces past the circumference of the le, σ being the mean outward radial velocity parallel to the sur- 's. Thus

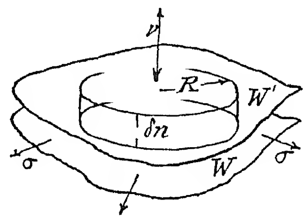


Fig 3

$$v = 2\sigma\frac{\delta n}{R}, \quad \ldots \quad \ldots \quad . \qquad \ldots\ldots (5)$$

he normal velocity is very small compared to the velocity parallel he surface.

In the case of a solid boundary it will be seen from the next ıgraph that the velocity σ is itself very small close to the surface, hat in this case the normal velocity v is a small quantity of the *nd* order.

Motion Parallel to Bounding Surfaces

With regard to the motion of fluids parallel to solid walls w which they are in contact, there is strong evidence that in the c of liquids at least the relative tangential velocity σ at the wall is ze Some of the evidence will be referred to later in connection with t flow of liquid through tubes under great pressure, and in the d cussion of the theory of lubrication.

Even when the mutual molecular attraction of a liquid and so appears to be comparatively small, so that the liquid does not tend spread over, or "wet" the surface of the solid, as is the case w mercury and glass, there is no observable sliding or slipping of t fluid over the solid at their common surface.

If the tangential tractional force between liquid and solid, a consequently the rate of shear in the liquid near the surface, finite, the relative tangential velocity, being zero at the surface, m be still small at all points of the liquid near the surface, as was serted in the last paragraph

In gases, the same rule as to the relative velocity being zero a solid surface is found to apply under ordinary circumstances, least as a very close approximation When, however, a gas is at su low pressure that its molecules are at distances apart comparable w the dimensions of the volume of gas which is being dealt wi phenomena are observed which can be regarded as arising from appreciable velocity of slipping of the gas over the solid surfa According to Maxwell,* the motion of the gas is very nearly the sa as if a stratum, of depth equal to twice the mean free path of the molecules, had been removed from the solid and filled with the g there being no slipping between the gas and the new solid surface.

At free surfaces, which, of course, can only exist in liquids, t normal velocity relative to the surface is again obviously zero. T liquid surface may, however, have a tangential velocity, and it usual to assume that the law of viscous shear holds up to the surfa and that either the tangential traction there becomes zero, or, if t liquid surface is exposed to a stream of air, that the traction is c only to the rate of shear in the air near the common surface. Expe ments by Rayleigh† and others have shown that, at least in the ca of water with an uncontaminated surface and of oils and other liqu which are capable of dissolving solid grease films, there are frictional resistances peculiar to the surface film

* *Collected Papers*, Vol II, p 708 † *Collected Papers*, Vol. III, p 363.

Viscous Flow in Tubes

On the principles which have been explained, we can proceed calculate the flow of viscous fluids in various cases which are practical interest Take first the case of a uniform tube of cular section of which the diameter is small compared to the gth of the tube. A fluid flows through the tube as the result a constant difference of pressure between its two ends. The tion, except very near the ends, will be sensibly parallel to the s of the tube, and the pressure (and consequently the density) l be sensibly uniform over every normal section. By symmetry, any one section the velocity must be the ne at all points at any given radius r from axis.

If w be the velocity (upwards in fig. 4) this radius, ρ the density, and p the ssure at any section, the radius of the e of the tube being a, the mass disrged per unit time, which must be the ie for all sections, is

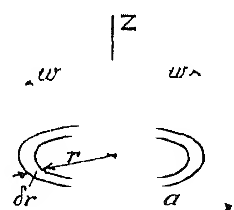


Fig 4

$$\int_0^a \rho w \, 2\pi r dr = m, \text{ constant} \quad .. (6)$$

e axis of the tube is taken as the axis Z, and is supposed to be so rly straight that effects due to its curvature can be neglected, and the first instance the motion will be supposed so slow that the etic energy of the fluid is inappreciable. The fluid may be either quid or gas. The effect of gravity is disregarded, or, if included, - $g\rho z$ is to be written instead of p.

From equations (4), p. 106, since the velocity w varies radially he rate $\frac{\partial w}{\partial r}$, but not circumferentially, there is a traction in the ction of Z on each unit of area of the cylindrical body of d inside radius r of amount

$$S_{rz} = \mu \frac{\partial w}{\partial r} \quad \ldots \quad (7)$$

nsidering a section of this cylinder of length δz, the total traction its cylindrical surface, whose area is $2\pi r \delta z$, must be equal to the

ifference of the total pressures on its upper and lower ends, so that

$$2\pi r S_{,z}\delta z = 2\pi r\mu\frac{\partial w}{\partial r}\delta z = \pi r^2\frac{dp}{dz}\delta z,$$

$$\text{or } \frac{\partial w}{\partial r} = \frac{r}{2\mu}\frac{dp}{dz},$$

$$\text{and therefore } w = \frac{1}{2\mu}\frac{dp}{dz}\left(\frac{r^2}{2} - C\right).$$

Since $w = 0$ when $r = a$, $C = \frac{a^2}{2}$, and

$$w = -\frac{1}{4\mu}\frac{dp}{dz}(a^2 - r^2). \quad . \quad (8)$$

ı which the negative sign expresses the obvious fact that the direc-
on of flow is opposite to the direction of increase of pressure.

Now from (6) and (8)

$$m = \int_0^a \rho w.2\pi r dr = -\int_0^a \frac{\rho}{4\mu} 2\pi r(a^2 - r^2)\frac{dp}{dz}dr$$

$$= -\frac{2\pi\rho}{4\mu}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^a\frac{dp}{dz}$$

$$= -\frac{\pi\rho a^4}{8\mu}\frac{dp}{dz}.^* \quad \ldots \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

In the case of a liquid, ρ and μ may usually be taken as constant, ɔ that $\frac{dp}{dz}$ is constant along the length of the tube, being equal to $\frac{{}_2 - p_1}{l}$ where p_1, p_2 are the pressures at the lower and uppeı ends f the tube whose length is l.

$$\text{Then } m = \frac{\pi a^4\rho}{8\mu}\frac{p_1 - p_2}{l} \quad \ldots\ldots\ldots\ldots \quad (10)$$

'he limits of application of this formula will be more fully explained ı a later chapter For the present it may be stated to be applicable) the flow of all liquids through "capillary" tubes (that is to say, ıbes whose diameter is only a fraction of a millimetre), unless the

* In all numerical applications of this and other formulæ throughout this ıapter all quantities must be expressed in the C.G.S. or other absolute system of ııts.

difference of pressures, $p_1 - p_2$, is greater than is ordinarily met with in engineering practice, provided that proper correction is made for the disturbing effects of the ends of the tube.

In the case of viscous lubricating oils, the formula is applicable, with certain restrictions, to their flow through ordinary lines of piping, but it must be regarded as subject to correction, or even wholly inapplicable, to the flow of the less viscous oils especially under considerable pressures.*

In the case of a gas, $\rho = \dfrac{p}{\mathrm{RT}}$, T being the absolute temperature and R a constant Thus from (9)

$$mdz = -\frac{\pi a^4}{8\mu \mathrm{RT}} pdp \quad \ldots \quad (11)$$

If T and μ can be regarded as constant throughout the length of the tube, integrating (11) we have

$$ml = \frac{\pi a^4}{16\mu \mathrm{RT}} (p_1^2 - p_2^2) \quad \ldots \quad (12)$$

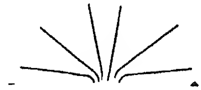
is the equation connecting the flow and the fall of pressure.

In the preceding discussion the kinetic energy of the fluid has been assumed to be negligible All the formulæ given, however, remain correct for the case of a liquid even when the kinetic energy is appreciable, provided that they are applied only to the middle portion of the tube and not to its end portions where the flow is affected by the acceleration and retardation of the fluid which occur near the inlet and outlet It is well known that the kinetic energy which a fluid acquires in entering an orifice is not wholly restored as pressure energy at its discharge There is therefore a resistance to the flow arising from the acceleration and retardation at the inlet and outlet of a tube, additional to the frictional losses within the tube itself. In the case of a square-ended tube opening into large vessels at each end, the loss of pressure is approximately $1{\cdot}12 \times \bar{u}^2/2g$, where $\bar{u}$ is the mean velocity at the outlet.†

There are further sources of resistance not taken into account in our calculations, arising from viscous friction between the streams at the ends, where the lines of flow are not parallel to the axis of the tube. Fig 5‡ shows the course of the particles of fluid at the inlet and outlet of a square-ended tube when the kinetic energy is appreciable and both ends of the tube are immersed in the fluid,

* See e.g. (13), p 159 † See Hosking, *Phil Mag*, April, 1909, Schiller, *Zeits Math. u. Mech*, Bond, *Proc. Phys. Soc*, 34, IV. ‡ From (10), p. 158.

Fig. 6 illustrates the condition which occurs when the outlet of such a tube is not immersed but discharges the fluid in a series of drops. In this case there is another resistance to the flow, due to the excess of internal pressure which is necessary to extend the surfaces of the drops during their formation.



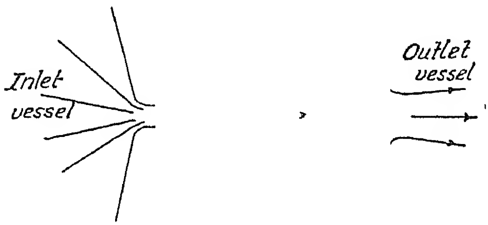


Fig 5

Fig 6

The calculation of the resistances due to these disturbing effects is rather uncertain, and on this account an accurate correspondence between the results of calculation and those of experiment can only be expected when the tubes are very long compared to their diameters

Use of Capillary Tubes as Viscometers

The experimental determination of coefficients of viscosity is carried out by instruments of various kinds, known as "viscometers" or "glischrometers" These are divided into two classes, namely "absolute" viscometers, by the use of which the coefficient of viscosity can be determined in absolute measure directly from the dimensions of the instrument itself (combined with measurement of a time interval), and "secondary" or "commercial" viscometers, which require to be calibrated by comparison of their results with those of an "absolute" viscometer.

The best absolute viscometers, for liquids at least, depend on the measurement of flow through capillary tubes, μ being determined from the equation (10) given on p. 110, after instrumental measurement of the other quantities involved. The apparatus by which Poiseuille made the first accurate determinations of the viscosity of water was of this class. The tubes which he used varied in diameter from 0·001 to 0·014 cm., their lengths being a few centimetres, and the pressure was applied by a column of mercury up to 77 cm. in height. Such instruments are capable of very considerable accuracy

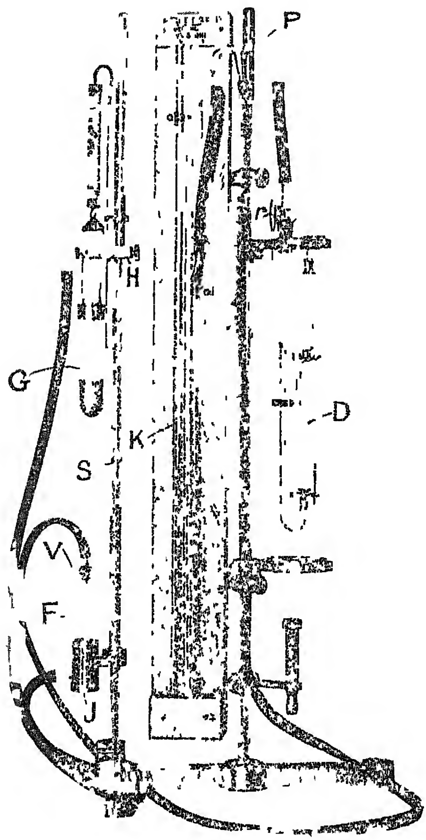


Fig 7—Stone's Absolute Viscometer

hen used with proper precautions, and when the necessary cor-
·ctions are applied for the various disturbing factors. The viscosity
` water, for instance, at atmospheric temperatures is probably known
ithin $\frac{1}{10}$th of 1 per cent of its true value.*

* See (11), p. 158.

The consistency to this order of accuracy of determinations made with different instruments and under different conditions is conclusive evidence of the correctness of the basic assumption of the linear connection of traction with shear, and of the absence of slipping of the fluid over the walls of tubes The principal precaution which has to be taken in the use of the capillary viscometer, in addition to the elimination of (or, in so far as that is not possible, the correction for) the end-disturbances which have been pointed out, is the accurate determination of the temperature of the fluid under test. The latter requirement is usually met by surrounding the capillary tube with a water-jacket, means being provided for warming or cooling the water, and measuring its temperature.

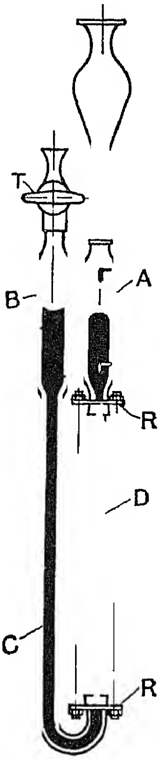


Fig 8

The most convenient form of "absolute" capillary viscometer for liquids is that described by W. Stone (18, p. 159). In this instrument the pressure is applied by a column of mercury of which the height is automatically maintained constant, and other devices are provided which further simplify the manipulation of the instrument and the calculation of the results from the observations The Stone viscometer is illustrated in fig. 7, the capillary tube and its attachments being shown separately in fig 8. The following is an abbreviation of the designer's description cited above.

The instrument consists of three essential elements, viz the viscometer burette, the adjustable constant-head apparatus, and the pressure-gauge. The viscometer burette consists of two glass vessels A and B (fig 8), of equal internal diameters and suitable lengths, connected at their lower ends by means of a wide-bore tube C, and of a capillary tube D of suitable dimensions for the desired purpose. The three portions of the burette are held together by the brass clips and tension-rods R. Several interchangeable tubes D may be provided for fluids of different viscosities.

The measuring vessel A is provided with two platinum wires sealed into its wall, and so bent that the inner end of each wire lies on the axis of the tube. The capacity of the vessel between the two platinum points can be thus accurately measured. A glass tap T

is provided on the inlet to the burette to control the starting of a test. The whole of the burette is immersed in water contained in a glass tube (see fig. 7) having a brass bottom. A brass cover is also fitted having a slot for the insertion of a stirring rod and a thermometer A Bunsen burner serves to heat the water.

The adjustable constant-head apparatus consists of two glass vessels, the lower one F being furnished with a tap V at the top and the upper one G suspended by a spring from a hook attached to a sliding clip H which can be clamped to the standard S at any desired height. Through the outer end of the clip a glass siphon pipe passes to the bottom of the vessel G when the latter is at its highest point, i.e against the clip H The siphon is connected to the lower vessel F by means of a rubber tube. The strength of the spring is so adjusted that as the mercury flows from G to F, the former, being thereby lightened, will rise so as to maintain the surface of the mercury in it at constant height above that of the mercury in F.

The pressure-gauge K is of the ordinary U pattern, with mercury as the working fluid A three-branch pipe P connects the burette, pressure-gauge, and constant-head apparatus.

The instrument must be set up vertical. As the liquid to be tested is fed into the burette A (fig 8), the vessel F is removed from the socket J and raised to a sufficient height above G to reduce the air-pressure in B (fig 8) and thus draw the liquid under test into it, lowering the surface in A below the lower platinum gauge-point. The glass tap T is then closed and the pressure apparatus adjusted to the desired pressure. Then the tap is opened, and the time elapsing between the moments of contact of the liquid surface with the gauge-points in A is taken by means of a stop-watch or suitable chronograph.

By the use of this instrument the viscosity of a sample of oil can be determined at ten or twelve different temperatures within an hour The pressure can be varied from about 5 to 50 cm of mercury in order to give (without changing the tube D) convenient intervals of time for measurement according to variations in the viscosity of the oil

Various other forms of apparatus have been used for the absolute determination of viscosities, their action depending, for instance, on the torsional oscillations of a disc or cylinder (a method which is convenient for measurement of the viscosity of gases, on account of the accuracy with which the very small forces involved may be

measured by this means), the continuous rotation of a cylinder or disc or sphere, or the free fall of a sphere in a body of fluid For general purposes, however, no other method is so convenient or accurate for absolute measurements of viscosity as that of Poiseuille.

Secondary or Commercial Viscometers

Tube viscometers are also commonly employed for making practical or commercial measurements of viscosity In order to reduce the time occupied by the measurements, and to simplify the apparatus and to reduce its delicacy, much shorter tubes are used in these instruments than are admissible for absolute instruments.

In the Redwood viscometer, for instance, the tube is approximately 1·7 mm. in diameter, and 12 mm in length, being a hole drilled through an agate plug fixed in the bottom of a vessel which is arranged to contain a measured quantity of the liquid to be tested. The liquid flows out of the hole under the force of gravity, the time of efflux of the measured quantity being taken by a stop-watch. Means are provided for warming or cooling the liquid to any temperature at which it is desired to make the test, but the determination of the actual temperature of the fluid as it is passing through the hole is one of the chief difficulties in the use of this and similar instruments.

In some of these the "tube" is so much reduced in length as to become a mere orifice. It will be readily understood that the corrections for the end effects of the tube, which have been pointed out as necessary in connection with all cases of viscous flow in tubes, become relatively much more considerable in the case of such short-tube instruments. In these, except for the more viscous liquids, the times of efflux are no longer proportional to the viscosity of the fluid. It is therefore necessary, in order to obtain reasonably accurate results, that such instruments should be calibrated over the range of their intended application by comparison with an absolute viscometer. Such a system of calibrations not having been generally adopted, an unfortunate practice has become common of expressing viscosities, not in terms of physical or engineering units (by which alone the value of the unit can be applied in calculations), but by the number of seconds or minutes required for the efflux of a certain volume through

ne or other of the best-known forms of commercial viscometers. There are thus in use as many arbitrary, irreconcilable, and dynamically meaningless units of viscosity as there are manufacturers of commercial viscometers.

A different type of secondary viscometer recently introduced is the cup-and-ball viscometer. The action of this instrument depends on the viscous flow of the fluid, not in a tube, but between two nearly parallel and closely adjacent surfaces. The instrument and its mode of operation will be more fully described below, after discussion of the theory of that type of viscous motion.

Coefficients of Viscosity of Various Fluids

In Table I (p. 118) are given values of the viscosity constant of a few of the fluids which are of chief interest to engineers, especially in connection with lubrication. The table contains also approximate numerical data, for the same fluids, of certain other physical properties, the significance of which, as affecting the utility of the fluids as lubricants, will be made more apparent by the later portions of this chapter. The constants are expressed in all cases in C G S units. The value of μ for instance is the ratio of a stress measured in dynes per square centimetre to a rate of shear measured in centimetres per second per centimetre.

The values of μ are given for various temperatures between 0° and 100° C. The other constants, which for the most part do not vary rapidly with temperature, are stated for atmospheric temperatures in the neighbourhood of 15° C. The rule which is apparent from the table as to the values of μ for liquids, namely that the value of each liquid diminishes as the temperature rises, is true generally. It will be noticed that the rate of variation is much less rapid for mercury and carbon bisulphide than for the other liquids. In all cases, as in air, on the other hand the viscosity increases with the temperature.

Variation of Viscosity with Pressure

The viscosity of both liquids and gases varies very little with variations of pressure over a range from many times less, to many times greater, than atmospheric pressure. At pressures, however, of the order of intensity of hundreds of atmospheres most liquids appear to have greatly increased coefficients of viscosity.

TABLE I

Coefficients of Viscosity, &c, of various Fluids in C G.S Units (at Atmospheric Pressure)

Fluid	Coefficient of Viscosity μ 0° C	20° C	40° C	60° C	80° C	100 °C	Surface Tension in Air	Density	Specific Heat	Thermal Conductivity	Freezing (F) or Setting Point (S)	Boiling (B) or "Flash" Point (F)
Water	0 0179	0 0101	0 0066	0 0048	0 0036	0 0028	75	1 00	1 00	13×10^{-4}	0°(F)	100° (B)
Mercury	0 0170	0 0157				0 0122	527	13 55	0 033	197×10^{-4}	− 39°(F)	357° (B)
Carbon bisulphide	0 0044	0 0038	0 0032	—	—	—	30	1 29	0 240	$3\ 4 \times 10^{-4}$	− 112°(F)	46° (B)
Glycerine	46	8 7					63	1·26	0 576	$6\ 4 \times 10^{-4}$	—	—
Sperm oil	1 13	0 335	0 173	0 079	0 040		38	0 878	0 493		0°(S)	150° (F)
Castor oil	—	7 24	2 23	0 68	0 28	0 12	38	0 969	0 508	$3\ 95 \times 10^{-4}$	− 15°(S)	205° (F)
Mineral Oils												
A "turbine-bearing oil"	—	1 528	0 245	0 110		0 035		0 88				
A "heavy" bearing oil ("Bayonne")	6 40	1 72	0 54	0 22	0 12	0 06	36	0 91	0 460	$3·5 \times 10^{-4}$	0°(S)	190° (F)
A "cylinder" oil ("Mobiloil BB")	—	9 47	1 30	0 54	0 26	0 12		0 91			3°(S)	185° (F)
Air	$1\ 73 \times 10^{-4}$	$1\ 86 \times 10^{-4}$	$2\ 01 \times 10^{-4}$	$2\ 1 \times 10^{-4}$	$2·2 \times 10^{-4}$	$2\ 3 \times 10^{-4}$	—	$1·29 \times 10^{-3}$	0 170 C_v 0 238 C_p	$5\ 68 \times 10^{-5}$	—	—

The following table (Table II) from Hyde (21, p. 159) shows how the viscosities of a few lubricating oils vary with pressures of this order Although such pressures do not usually exist in ordinary bearings, there are cases in the application of the theory of viscosity, as will be seen later in this chapter, in which the changes of the viscous constant by increase of pressure cannot be neglected.

Within very wide limits, the viscosity of gases is independent of pressure, the viscosity of air for instance being practically invariable from a pressure of a few millimetres of mercury up to pressures of many atmospheres This law, originally predicted by Maxwell from the kinetic theory of gases, has been confirmed by numerous experiments

TABLE II

VISCOSITIES OF VARIOUS LUBRICATING OILS AT VARYING PRESSURES. TEMPERATURE 40° C

Abstracted from a table by J H Hyde *Proc Roy Soc*, A 97

Pressure, Kilograms per Square Centimetre	Mineral Oil ("Bayonne")	Trotter Oil (Animal)	Rape Oil (Vegetable)	Sperm Oil
	Coefficient of Viscosity, μ, C G S			
0	0 47	0 344	0 375	0 154
157·5	0 62	0 413	0 422	0 190
315 0	0 92	0 550	0 539	0 236
472 5	1 32	0 686	0 703	0 299
630 0	1 86	0 824	0 880	0 368
787 5	2 51	—	1 089	—
945	3 65	1 217	1 310	—
1102 5	5 32	—	1 578	0 619
1260	7 55	1 731	—	—

Viscous Flow between Parallel Planes

As one of the typical conditions of flow met with in problems of lubrication and other practical applications of the theory of viscosity, it is convenient to consider in detail the flow of viscous liquid between two parallel and closely adjacent plane walls, supposed fixed

In rectangular co-ordinates, let $z = 0$, and $z = h$ be the parallel planes, h being small compared to their dimensions in the X and Y directions, as indicated in fig 9

For the reasons already explained the components of velocity

normal to the planes must be everywhere negligible In other words, the rates of shear and the momentum in the Z direction are very small; consequently the fluid pressure p does not vary in that direction but is a function of x and y only. Also the rates of change from the values of the finite velocity components, u, v, in the fluid to their values, known to be zero, on the walls are rapid compared to their rates of change in the X and Y directions. Thus, considering a rectangular element, as in fig 10, anywhere between the

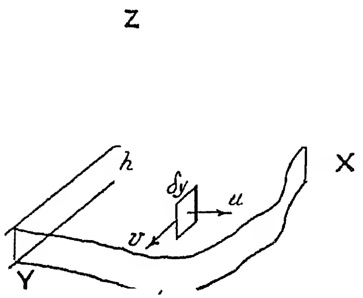


Fig 9

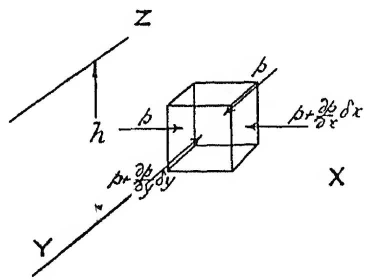


Fig 10

planes $z = 0$ and $z = h$, the viscous tractions on its lower face in the directions in which x and y increase, are.

$$-\mu\frac{\partial u}{\partial z}\delta x\delta y,$$

$$\text{and}\quad -\mu\frac{\partial v}{\partial z}\delta x\delta y$$

The corresponding tractions on the upper face are

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial z^2}\delta z\right)\delta x\delta y,$$

$$\text{and}\quad \mu\left(\frac{\partial v}{\partial z}+\frac{\partial^2 v}{\partial z^2}\delta z\right)\delta x\delta y.$$

The sums of these pairs of tractions added to the differences of the fluid pressures on the faces parallel to the YZ and ZX planes are respectively equal to the rates of increase of the momentum of the element in the X and Y directions, thus

$$\left(\mu\frac{\partial^2 u}{\partial z^2}-\frac{\partial p}{\partial x}\right)\delta x\delta y\delta z = \rho\delta x\delta y\delta z\frac{du}{dt},$$

$$\left.\begin{aligned} \text{or} \quad \mu\frac{\partial^2 u}{\partial z^2} &= \frac{\partial p}{\partial x} + \rho\frac{du}{dt}, \\ \text{and similarly} \quad \mu\frac{\partial^2 v}{\partial z^2} &= \frac{\partial p}{\partial y} + \rho\frac{dv}{dt}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (13)$$

ρ being the density of the liquid

The rates of increase of velocity $\frac{du}{dt}$, $\frac{dv}{dt}$ are of the order of the products $u\frac{\partial u}{\partial x}$, and $v\frac{\partial v}{\partial y}$, and are thus, if, as we assume, u and v are small, of the order of squares of small quantities. These momentum terms will therefore be neglected in this and the following discussions. With this stipulation the equations (13) reduce to

$$\left.\begin{aligned} \mu\frac{\partial^2 u}{\partial z^2} &= \frac{\partial p}{\partial x}, \\ \text{and} \quad \mu\frac{\partial^2 v}{\partial z^2} &= \frac{\partial p}{\partial y}. \end{aligned}\right\} \quad \ldots\ldots\ldots (14)$$

These can be directly integrated, since p is independent of z, and thus

$$\frac{\partial u}{\partial z} = \frac{1}{\mu}\frac{\partial p}{\partial x}(z + C_1),$$

$$\text{and} \quad u = \frac{1}{\mu}\frac{\partial p}{\partial x}\left(\frac{z^2}{2} + C_1 z + C_2\right),$$

$$\text{and similarly} \quad v = \frac{1}{\mu}\frac{\partial p}{\partial y}\left(\frac{z^2}{2} + D_1 z + D_2\right)$$

Now since u and v are zero on the plane $z = 0$, the integration constants C_2 and D_2 are each zero, and since u and v are also zero on the plane $z = h$:

$$\frac{h^2}{2} + C_1 h = \frac{h^2}{2} + D_1 h = 0,$$

$$\text{so that} \quad C_1 = D_1 = -\frac{h}{2}.$$

$$\text{Thus} \quad u = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{z(z-h)}{2}, \quad \ldots\ldots\ldots (15)$$

$$\text{and} \quad v = \frac{1}{\mu}\frac{\partial p}{\partial y}\frac{z(z-h)}{2}, \quad \ldots\ldots\ldots\ldots (15a)$$

and the resultant velocity of the fluid at any point is

$$(u^2 + v^2)^{\frac{1}{2}} = \frac{1}{\mu}\left\{\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2\right\}^{\frac{1}{2}} \frac{z(z-h)}{2}, \quad \ldots\ldots(16)$$

being in the direction of, and proportional to, the most rapid fall of pressure, and varying according to a parabolic law along each normal from one plane to the other, having its maximum value midway between them.

The total flow across a width ∂y (see fig 9) from plane $z = 0$ to plane $z = h$ (in the direction of x increasing) is

$$\delta y \mathrm{U} = \delta y \int_0^h u dz = \frac{\delta y}{2\mu}\frac{\partial p}{\partial x}\int_0^h (z^2 - zh) dz$$

$$= \frac{\delta y}{2\mu}\frac{\partial p}{\partial x}\left[\frac{z^3}{3} - \frac{z^2 h}{2}\right]_0^h = -\delta y \frac{h^3}{12\mu}\frac{\partial p}{\partial x},$$

$$\text{or} \quad \mathrm{U} = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}. \quad \ldots\ldots(17)$$

Similarly the flow per unit width in the y direction

$$\text{is} \quad \mathrm{V} = -\frac{h^3}{12\mu}\frac{\partial p}{\partial y}. \quad \ldots\ldots(18)$$

Thus the total flow in any direction across a unit width perpendicular to that direction is equal to the rate of *decrease* of pressure in that direction multiplied by the constant $\frac{h^3}{12\mu}$

The same relation evidently holds for the flow of a viscous liquid in the space between two concentric, fixed cylinders, in either the axial or the circumferential direction, provided that the radii of the cylinders are so nearly equal that their difference can be neglected compared with either of them.

In both of these cases, as well as in all other cases of flow between parallel surfaces plane or curved, it is evident, considering any small rectangular element δx, δy, which extends in the z direction from one surface to the other, that since the same amount of fluid must flow out of, as flows into, the element in unit time,

$$h\left(\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y}\right)\delta x \delta y = 0,$$

or from (17) and (18) $$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0, \quad \ldots\ldots(19)$$

it being remembered that the surfaces $z = 0$ and $z = h$, are assumed to be fixed.

Flow between Parallel Planes having Relative Motion

If the plane $z = h$ is moving parallel to the plane $z = 0$, with components of velocity u_1 and v_1 in the X and Y directions, uniform rates of shear $\frac{u_1}{h}$ and $\frac{v_1}{h}$ in these two directions will be superimposed on the fluid velocities u and v of (15) and (15a). The components of velocity at z will become

$$u' = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{z(z-h)}{2} + u_1\frac{z}{h},$$

$$\text{and} \qquad v' = \frac{1}{\mu}\frac{\partial p}{\partial y}\frac{z(z-h)}{2} + v_1\frac{z}{h},$$

but neither the pressures nor the relation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

will be affected

If, on the other hand, the plane $z = h$ is caused to move normally away from the plane $z = 0$, with velocity $\frac{dh}{dt}$, so that the distance between the planes continually increases at this rate, it is evident that an excess of inflow over outflow must take place through the sides (at right angles to the planes) of the elementary volume $h\delta x\delta y$ to supply the additional volume which is continually being added to the element at the rate $\frac{dh}{dt}\delta x\delta y$

Expressing this equality in symbols,

$$\delta y\frac{\partial \mathrm{U}}{\partial x}\delta x + \delta x\frac{\partial \mathrm{V}}{\partial y}\delta y = -\frac{dh}{dt}\delta x\delta y,$$

$$\text{or} \qquad \frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} = -\frac{dh}{dt},$$

and consequently from (17) and (18),

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{12\mu}{h^3}\frac{dh}{dt}. \quad \ldots\ldots\ldots\ldots\ldots(20)$$

If we take the planes as being circular, of radius a, and suppose that the fluid between them at this radius is in direct communication with a large volume of the same fluid at constant pressure Π, it is evident from symmetry that the flow will be everywhere radially inwards and that the pressure will diminish from Π at radius a to a minimum at the centre. Taking, instead of a rectangular element, a cylindrical element extending from one plane to the other and contained between radii r and $r + \delta r$ as well as between two radial planes at a small angle $\delta\alpha$ apart, its rate of increase of volume, see fig. 11, will be $\frac{dh}{dt}\delta r . r\delta\alpha$. This must be equal to the rate of increase of the inward radial flow as r increases by δr, so that from (17), p. 123,

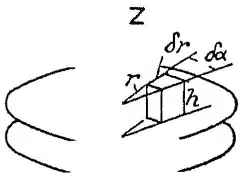


Fig 11

$$\frac{dh}{dt}\delta r\, r\delta\alpha = \frac{\partial}{\partial r}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial r} r\delta\alpha\right)\delta r,$$

$$\text{or} \quad \frac{12\mu}{h^3}\frac{dh}{dt}r = \frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right).$$

Integrating,

$$\frac{6\mu}{h^3}\frac{dh}{dt}r^2 = r\frac{\partial p}{\partial r} + \mathrm{C}.$$

But from (17), since the radial velocity is zero at the centre,

$$\frac{\partial p}{\partial r} = 0 \text{ when } r = 0,$$

$$\mathrm{C} = 0 \text{ and } \frac{\partial p}{\partial r} = \frac{6\mu}{h^3}\frac{dh}{dt}r,$$

so that integrating again

$$p = \frac{3\mu}{h^3}\frac{dh}{dt}r^2 + \mathrm{C}_1.$$

$$\text{When} \quad r = a, \; p = \Pi = \frac{3\mu}{h^3}\frac{dh}{dt}a^2 + \mathrm{C}_1,$$

$$\text{so that} \quad p = \Pi - \frac{3\mu}{h^3}\frac{dh}{dt}(a^2 - r^2). \qquad \ldots\ldots(21)$$

The force, P, necessary to move the plane at $z = h$, against the

viscous resistance is equal and opposite to the difference of pressure $p - \Pi$ integrated over the whole circle, or

$$\begin{aligned} P &= \int_0^a 2\pi r \frac{3\mu}{h^3}\frac{dh}{dt}(a^2 - r^2)dr \\ &= \frac{6\pi\mu}{h^3}\frac{dh}{dt}\int_0^a (a^2 r - r^3)dr \\ &= \frac{6\pi\mu}{h^3}\frac{dh}{dt}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) \\ &= \frac{3\pi\mu a^4}{2h^3}\frac{dh}{dt} \quad \text{......} \quad \text{..............(22)} \end{aligned}$$

Cup-and-ball Viscometer

The type of viscous motion which has just been discussed is that on which is based the action of the cup-and-ball viscometer already mentioned on p. 19, and illustrated in figs 12 and 13. In the actual instrument, however, as illustrated in fig. 12, the two parallel surfaces which are drawn apart are not planes but segments of two spheres, one concave and the other convex. The fixed surface is the concave lower surface of a metal cup, to which is attached a hollow handle by which the instrument is suspended In the cup fits a steel ball, but its surface is prevented from making actual contact with the spherical surface of the cup by three very small projections (J, fig. 12) from the cup's spherical surface The two spherical surfaces are thus maintained parallel and about 0 01 mm apart, when the ball rests on the projections The narrow interspace is filled with the liquid to be tested, and in addition a groove G formed around the edge of the cup, and having a capacity of a few cubic millimetres, is also filled with the fluid, which is held in both the groove and the interspace by capillary tension The groove forms the reservoir at constant pressure Π from which the interspace is fed with fluid when the two surfaces are drawn apart, as in the preceding calculations.

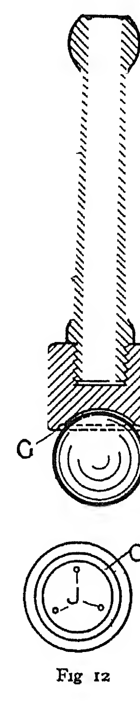


Fig 12

The force P employed to draw the surfaces apart is the weight

of the ball, which is usually of steel and 1 inch in diameter The method of making a test is merely, after placing sufficient liquid in

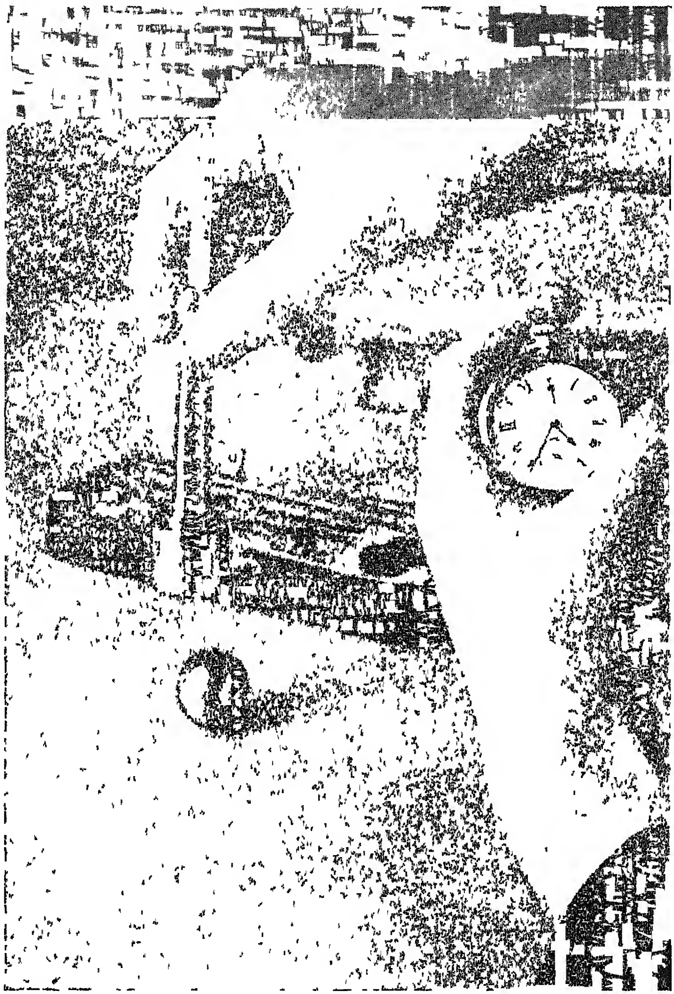


Fig 13 —Cup-and-ball Viscometer

the cup to fill the groove and interspace, and pressing the ball home, to suspend the whole instrument and note the time by stop-watch which the ball takes to detach itself. The temperature of the instrument, which, on account of the good conductivity of the

netal and the very small mass of liquid, is also very nearly the emperature of the latter, is observed at the same time by means of thermometer inserted in the hollow handle.

The time of fall, the dimensions of the instrument being given, an be calculated approximately from formula (22), the spherical egments concerned, which in the actual instrument are comparaively flat, being treated as circular planes of the same area.

$$\text{Since } \frac{dt}{dh} = \frac{3\pi\mu a^4}{2h^3 P},$$

$$t = \int_{h_1}^{h_2} \frac{3\pi\mu a^4}{2P}\frac{dh}{h^3} = \frac{3\pi\mu a^4}{4P}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right),$$

n which t is the time of fall of the ball, of weight P, from its initial listance h_1 to a final distance h_2 from the surface of the cup.

This fall is to be considered to be complete when the volume of luid drawn into the interspace is equal to the volume initially conained in the groove, i e

$$\pi a^2(h_2 - h_1) = 2\pi a S$$

where S is the sectional area of the groove,

$$\text{thus } h_2 = h_1 + \frac{2S}{a},$$

nd the time of the complete fall is

$$t = \frac{3\pi\mu a^4}{4P}\left\{\frac{1}{h_1^2} - \frac{1}{\left(h_1 + \frac{2S}{a}\right)^2}\right\}$$

$$= \frac{3\pi\mu a^4}{P}\frac{S(ah_1 + S)}{h_1^2(ah_1 + 2S)^2},$$

nd if S is large compared to ah_1, as it should be,

$$t = \frac{3\pi\mu a^4}{4Ph_1^2}, \quad \text{and} \quad \mu = \frac{4Ph_1^2 t}{3\pi a^4}. \qquad \ldots\ldots.(23)$$

It will be seen from the formula (22) that the velocity of fall $\frac{h}{t}$ varies as the cube of the distance fallen through. It is thus very mall at first, but increases very rapidly in the later stages, and there s no difficulty in practice in deciding the moment when the fall is irtually complete.

Although the action of the cup-and-ball viscometer can be calculated with sufficient accuracy when its dimensions, including the initial thickness of the fluid film, are known, the determination of this thickness, that is to say the height of the three projections in the cup, with sufficient accuracy would be so difficult that in practice the instrument is employed as a secondary viscometer only. Each instrument requires, however, only a single calibration test, which suffices to determine a single constant for the instrument, applicable over its whole range. The corrections for the momentum of the fluid and for capillarity are negligible, the former because the velocity of the fluid is exceedingly low and the latter because the radius of curvature of the meniscus of the liquid in the groove is very large compared to the thickness of the liquid film subject to viscous traction.

B. LUBRICATION

The Connection between Lubrication and Viscosity

Although viscous liquids and plastic solids have been used from the earliest times to diminish friction between solid bodies moving in contact with one another, and although the practice of thus "lubricating" the bearings of machines has doubtless been universal since machines were first constructed, no rational explanation of the action of the lubricant was known until Osborne Reynolds (5), in 1886, gave a clear interpretation of the phenomena in terms of the theory of viscosity. Reynolds' explanation was only complete in a quantitative sense in the case of journal bearings furnished with special, and at that date unusual, means for supplying ample quantities of lubricant. He showed that in such cases the solid surfaces are completely separated from one another by fluid films of appreciable thickness, and that such films are maintained and enabled to support the pressure imposed on them quite automatically by the relative motion of the parts. The theory has since been extended to bearings of other kinds than journal bearings, and by its application new types of bearings have been devised for various purposes which have proved far more efficient than the forms which they were designed to replace.

While this "viscosity theory" of bearing lubrication is not quantitatively complete in all cases, and while there are probably other modes of lubrication in which viscosity does not play an essential part, it is at present true that all the most efficient known

ypes of bearings which operate with sliding, as distinguished from olling, contact utilize the principle of lubrication which was dis- :overed by Reynolds. The experimental and theoretical work by vhich the principle has been developed may be followed in the papers quoted in the bibliography attached to the end of this chapter. t is only possible in the present chapter to give an outline of the heory and a few of the leading results which have been established, vith examples of the practical forms of bearings in which the theory ias been utilized

The feature common to all the bearings to which Reynolds' heory can be applied is that the surfaces of the relatively moving parts are not exactly parallel but slightly inclined to one another. For instance, in order that a journal bearing of the usual type may e lubricated according to Reynolds' principle, it is necessary that he journal shall be slightly eccentric in the bearing, so that the ilm of lubricant shall be of a thickness varying around the ournal

Similarly, for the proper lubrication of a slipper moving rela- ively to a plane surface, it is necessary that the surface of the lipper, if plane, shall be slightly inclined to the plane surface ver which it moves.

Essential Condition of Viscous Lubrication

The explanation of this essential condition is readily given as n extension of the calculations contained in the first part of this hapter.

Usually in the bearings to which the theory is applicable one of he surfaces can be considered as ontinuous or unlimited in dimen- ion in the direction of the relative notion (as for instance the surface f a journal, or a thrust collar, or n engine cylinder), while the sur- ace of the other member is essen- ially limited or discontinuous in he same direction (as the surfaces f the corresponding bearing-brass,

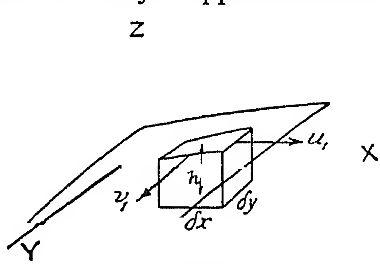


Fig 14

hrust-bearing shoe, or engine piston). In fig. 14, let XY be axes f co-ordinates (straight or curved) in directions at right angles to ach other along the surface of the continuous element, and Z the

co-ordinate axis normal to this surface, i e. in the direction of the thickness of the film, and as before let u, v, w be the components of the velocity of the fluid at any point in these three directions The surfaces of the continuous and discontinuous elements are assumed to be nearly parallel, and the distance between them h to be small compared to their radii of curvature. The discontinuous surface is supposed to move with components of velocity u_1, v_1 in the X and Y directions, parallel to the continuous surface, at xy. The problem of finding the motions and pressures of a viscous film between the surfaces is the same as that discussed on p 123, except that the surfaces are not now parallel. Considering, as before, the rate of change of volume and the flow of fluid into and out of an element extending from one surface to the other and standing on the base δx, δy, it is seen from equations (17), (18), that the rate of increase of volume of fluid in the element due to the rates of change of pressure and of film-thickness, in the X and Y directions is

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right)\delta x\delta y + \frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\right)\delta y\delta x, \quad \ldots\ldots \quad (24)$$

while the rate at which fluid passes out of the element in consequence of the shearing deformation due to the movement of the upper surface over the lower is

$$\frac{u_1}{2}\frac{\partial h}{\partial x}\delta x\delta y + \frac{v_1}{2}\frac{\partial h}{\partial y}\delta y\delta x. \quad \ldots \quad \ldots\ldots (25)$$

The volume of the element is however diminishing, in consequence of the movement of the upper plane, at the rate

$$u_1\frac{\partial h}{\partial x}\delta x\delta y + v_1\frac{\partial h}{\partial y}\delta y\delta x,$$

consequently

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\right) - \left(\frac{u_1}{2}\frac{\partial h}{\partial x} + \frac{v_1}{2}\frac{\partial h}{\partial y}\right) = -\left(u_1\frac{\partial h}{\partial x} + v_1\frac{\partial h}{\partial y}\right),$$

or $$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right) + 6\mu\left(u_1\frac{\partial h}{\partial x} + v_1\frac{\partial h}{\partial y}\right) = 0 \quad (26)$$

This is the general differential equation determining the value of p at every point, being solved by integration for each particular case when h is given as a function of x and y (thus defining the forms of the surfaces), and when the velocities u_1, v_1 are assigned. The complete solution is often not practicable, but exact or approximate

lutions can be obtained in a number of the simpler cases which n be regarded as sufficiently close approximations to the actual nditions of various types of bearings.

Inclined Planes Unlimited in one Direction

Take the case of two plane surfaces, the lower, $z = 0$, being ılımited in the directions of X and Y, while the upper, also un- nited in the direction of Y,* extends only from $x = a_1$, to $x = a_2$, d intersects the plane $z = 0$ on the line $x = 0$. Thus the dis- nce between the planes, erywhere small, is propor- nal to x, so that $h = cx$, iere c is the tangent of the iall angle between the planes e fig. 15). Let us assume at the upper plane moves er the lower with velocity , in the direction of X, v ing zero, and that the iole is immersed in fluid, so that the pressure both in front and behind the moving plane is Π and is constant. Ob- ously none of the conditions vary in the direction of Y, so that and $\dfrac{\partial p}{\partial y}$ are both zero. Thus equation (26) becomes

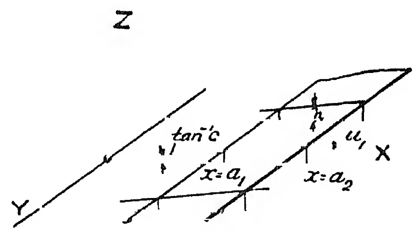


Fig 15

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + 6\mu u_1\frac{\partial h}{\partial x} = 0,$$

and therefore $$h^3\frac{\partial p}{\partial x} + 6\mu u_1(h - h_1) = 0, \quad\ldots\ldots\ldots\ldots(27)$$

being the value of h where $\dfrac{\partial p}{\partial x} = 0$, that is to say at a point, x_1, where p has a maximum or minimum value

$$\text{Thus}\quad \frac{\partial p}{\partial x} = -6\mu u_1\left(\frac{1}{h^2} - \frac{h_1}{h^3}\right) = -\frac{6\mu u_1}{c^2}\left(\frac{1}{x^2} - \frac{x_1}{x^3}\right).$$

ıce $\dfrac{\partial p}{\partial x}$ is positive when $h < h_1$, and negative when $h > h_1$, it is seen

* The dimension of a bearing in the direction of the motion will in all cases be erred to as its length, and the transverse dimension as its width, regardless of ich of these is the greater.

that p has a maximum value (at $x = x_1$), between $x = a_1$, and $x = a_2$.

Integrating, $$p = \frac{6\mu u_1}{c^2}\left(\frac{1}{x} - \frac{x_1}{2x^2} - C\right), \qquad (28)$$

but since $p = \Pi$, both when $x = a_1$, and when $x = a_2$,

$$\Pi = \frac{6\mu u_1}{c^2}\left(\frac{1}{a_1} - \frac{x_1}{2a_1^2} - C\right) = \frac{6\mu u_1}{c^2}\left(\frac{1}{a_2} - \frac{x_1}{2a_2^2} - C\right),$$

from which

$$\frac{x_1}{2}\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right) = \left(\frac{1}{a_1} - \frac{1}{a_2}\right),$$

$$\text{or} \qquad x_1 = \frac{2a_1a_2}{a_1 + a_2} \qquad (29)$$

(the point of maximum pressure thus being nearer to a_1 than to a_2),

$$\text{and} \quad -\frac{6\mu u_1 C}{c^2} = \Pi - \frac{3\mu u_1}{c^2}\left\{\frac{1}{a_1} + \frac{1}{a_2} - \frac{x_1}{2}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)\right\}$$

$$= \Pi - \frac{6\mu u_1}{c^2(a_1 + a_2)}.$$

Thus by substitution for x_1, and C in (28),

$$p = \Pi + \frac{6\mu u_1}{c^2}\left(\frac{1}{x} - \frac{a_1a_2}{(a_1 + a_2)x^2} - 1\right)$$

$$= \Pi + \frac{6\mu u_1}{c^2(a_1 + a_2)}\left(\frac{a_1 + a_2}{x} - \frac{a_1a_2}{x^2} - 1\right), \quad \ldots \quad (30)$$

this equation determining the pressures at all points between the two planes.

The total upward pressure on the upper plane per unit width in the direction Y is

$$P = \int_{a_1}^{a_2}(p - \Pi)dx = \frac{6\mu u_1}{c^2(a_1 + a_2)}\int_{a_1}^{a_2}\left(\frac{a_1 + a_2}{x} - \frac{a_1a_2}{x^2} - 1\right)dx$$

$$= \frac{6\mu u_1}{c^2(a_1 + a_2)}\left[(a_1 + a_2)\log_e x + \frac{a_1a_2}{x} - x\right]_{a_1}^{a_2}$$

$$= \frac{6\mu u_1}{c^2(a_1 + a_2)}\left\{(a_1 + a_2)\log_e\frac{a_2}{a_1} + 2(a_1 - a_2)\right\}$$

$$= \frac{6\mu u_1}{c^2}\left\{\log_e\frac{a_2}{a_1} + 2\frac{a_1 - a_2}{a_1 + a_2}\right\}, \qquad (31)$$

eing dependent only on the ratio of a_2 to a_1, for a given value of c, nd the mean pressure is

$$\frac{\mathrm{P}}{a_2 - a_1} = \frac{6\mu u_1}{c^2}\left\{\frac{1}{a_2 - a_1}\log_e\frac{a_2}{a_1} - \frac{2}{a_2 + a_1}\right\} \ldots\ldots(32)$$

lso the total frictional resistance to the motion of the upper plane er unit width is

$$\mathrm{F} = \int_{a_1}^{a_2}\mu\frac{u_1}{h}dx = \int_{a_1}^{a_2}\frac{\mu u_1}{cx}dx \ \ldots \quad .. \quad .(33)$$

$$= \frac{\mu u_1}{c}\log_e\frac{a_2}{a_1}, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33a)$$

ependent, like P, only on the ratio $a_2 : a_1$ and c; and the ratio f traction to load, or " coefficient of friction " is

$$f = \frac{\mathrm{F}}{\mathrm{P}} = \frac{c}{6}\,\frac{\log_e\dfrac{a_2}{a_1}}{\log_e\dfrac{a_2}{a_1} - 2\dfrac{a_1 - a_2}{a_1 + a_2}} \quad . \quad . \qquad (34)$$

lso the position of the centre of the upward pressure on the upper 'ane is given by

$$\bar{z} = \frac{1}{\mathrm{P}}\int_{a_1}^{a_2}(p - \Pi)x\,dx = \frac{6\mu u_1}{\mathrm{P}c^2(a_1 + a_2)}\int_{a_1}^{a_2}\left\{(a_1 + a_2) - \frac{a_1 a_2}{x} - x\right\}dx$$

$$= \frac{6\mu u_1}{\mathrm{P}c^2(a_1 + a_2)}\left\{\frac{a_1^2 - a_2^2}{2} - a_1 a_2\log_e\frac{a_2}{a_1}\right\}$$

$$= \frac{1}{2}\,\frac{a_2^2 - a_1^2 - 2a_1 a_2\log_e\dfrac{a_2}{a_1}}{2(a_1 - a_2) + (a_1 + a_2)\log_e\dfrac{a_2}{a_1}}, \ \ldots \qquad (35)$$

ing independent of c.

Applications to Actual Bearings

The solution of the problem in viscous motion illustrated in . 15 has been worked out in some detail because it affords in a igle case a general view of the nature of Reynolds' theory of lubrition.

If we imagine the lower plane $z = 0$ replaced by the surface of ylinder whose axis is parallel to the Y axis of co-ordinates, and

the upper plane, extending from $x = a_1$, to $x = a_2$, replaced by a curved surface which, at every point of co-ordinates x, y, measured respectively circumferentially from a generating line of the cylinder corresponding to $x = 0$, and axially from a circumferential circle of the cylinder corresponding to $y = 0$, is at the same normal distance h from the cylinder as are the two planes from one another, the results which have been obtained will still apply This ideal form of a cylindrical journal bearing is illustrated in fig. 16 The cylinder can be regarded as the journal of an axle, and the upper surface as the bearing surface of the bearing-brass of the axle.

The results as to the fluid pressure which have been calculated above evidently remain true if, instead of the bearing-brass moving in the direction x with linear velocity u_1, the journal revolves in the opposite direction with the same surface velocity.

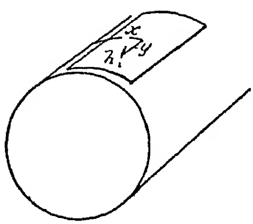


Fig 16

Actual bearings are, of course, not of unlimited width, but for the middle portions of a bearing whose dimension in the direction transverse to the relative motion is not less than two or three times that in the direction of motion, the calculated results apply with fair accuracy. In such middle portions of the bearing the oil will flow in lines approximately at right angles to the generating lines of the cylinder. In the lateral portions of the bearing, on the other hand, the oil being under pressure will tend to flow towards the nearest side, and the theoretical conditions will on this account be departed from If, however, the sides of the bearing be closed by some arrangement, such as a stuffing-box, preventing the escape of oil, the flow of oil will be everywhere, except within distances from the closed sides comparable with h_1, circumferential, and the conditions assumed for unlimited surfaces will be precisely realized, provided always, of course, that the bearing-brass is of such a form that $h = cx$, which is true only to a first approximation for the form which is usually given to such brasses.

The calculations apply more accurately to the case of a conical sleeve moving longitudinally on a cylindrical rod as illustrated in fig. 17. In this figure the axis of X is a generating line of the cylindrical surface of the rod, the axis of Y is a circumferential circle, and that of Z as before is normal to the surface. As before, we assume that the normal distance between the surfaces is given by $h = cx$,

that the conical and cylindrical surfaces, which are coaxial, inter-ct at $x = 0$. The sleeve extends from $x = a_1$ to $x = a_2$, and supposed to move parallel to the axis of X with velocity u_1.

From symmetry the motion of the fluid must be everywhere rallel to the axis of X, and as the cone and film of fluid have no undaries in the direction of Y, the solution given above will hold curately provided that the thickness of the film is very small mpared to the radius r_1, and length, $a_2 - a_1$, of the cone. Thus, for ample, the resistance to the motion of the cone, from (33a), p. 133, is

$$2\pi r_1 F = \frac{2\pi r_1 \mu u_1}{c} \log_e \frac{a_2}{a_1}$$

ıe curve p_1 in fig. 18 shows the mode in which the fluid pressure

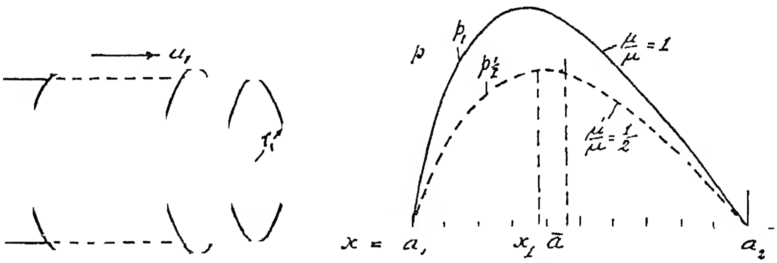


Fig 17 Fig 18

ween the surfaces of figs 15, 16, and 17 varies in the direction x for the particular case in which $a_2 = 2a_1$. It will be seen that maximum pressure occurs at $x_1 = \frac{4}{3}a_1$, or at one-third of the gth of the sleeve or bearing-brass from its rear end, and, as may seen by writing $2a_1$ for a_2 in (35), the resultant pressure urs at $\bar{a} = 1{\cdot}431a_1$, or 0·431 of the length from the same end.

Table III, p. 130, shows the actual numerical results in G.S units for a moving surface carrying a resultant pressure of Kgm. with a lubricating fluid of viscosity 1 C G.S The surface issumed to be 1 cm long in the direction of motion (i e. $a_2 - a_1$ 1 cm), and the results are expressed for 1 cm. of width in the nsverse direction The quantities tabulated are·

h_1, the thickness of film at $x = a_1$, unit 10^{-3} cm.;
h_2, the thickness of film at $x = a_2$, unit 10^{-3} cm.;

- a_1, the distance of the centre of pressure from the trailing end.

Unit, 1 cm ; f, the effective coefficient of friction, $= F_1$ the tractive force in kilograms.

The independent variable in the first column of the table is the ratio $\frac{a_2}{a_1} = \frac{h_2}{h_1}$.

TABLE III

$\frac{a_2}{a_1}$	h_1.	h_2	$\bar{a} - a_1$.	f.
1·0	0	0	0·5000	∞
1·2	0·2775	0·3330	0·4818	$3·285 \times 10^{-3}$
1·4	0·3465	0·4851	0·4664	2·428
1·6	0·3793	0·6079	0·4532	2·065
1·8	0·3955	0·7119	0·4416	1·858
2·0	0·4026	0·8051	0·4313	1·722
2·2	0·4043	0·8895	0·4221	1·625
2·4	0·4027	0·9665	0·4137	1·553
2·6	0·3991	1·0375	0·4061	1·497
2·8	0·3942	1·1037	0·3991	1·451
3·0	0·3884	1·1652	0·3926	1·414
4·0	0·3559	1·4237	0·3662	1·298
5·0	0·3247	1·6237	0·3465	1·239
6·0	0·2982	1·7892	0·3310	1·202
11·0	0·2115	2·3269	0·2832	1·134

The corresponding results for any other dimensions and conditions of loading may be derived from the following dimensional formulæ, viz :

If the length of the surface, velocity, resultant load, and viscosity, instead of being each unity in the units employed, are respectively

Length, L centimetres,
Velocity, V centimetres per second,
Load, P kilograms per unit width,
Viscosity, M C.G.S. units,

then, for any given value of $\frac{a_2}{a_1}$, h_1 and h_2 are to be multiplied by $\frac{LV^{\frac{1}{2}}M^{\frac{1}{2}}}{P^{\frac{1}{2}}}$, $\frac{\bar{a} - a_1}{a_2 - a_1}$ is unchanged, and F is to be multiplied by $V^{\frac{1}{2}}P^{\frac{1}{2}}M^{\frac{1}{2}}$, while c and f are to be multiplied by $\frac{M^{\frac{1}{2}}V^{\frac{1}{2}}}{P^{\frac{1}{2}}}$.

It will be seen from Table III, combined with these dimensional formulæ, that the thicknesses of the films of viscous fluid concerned

ı lubrication are small, and comparable to the smallest linear mea-ırements which the mechanical engineer is accustomed to make. t is therefore necessary in order to effect lubrication in the manner ıtended, and to secure the low frictional resistances which the ıeory indicates as attainable, that the workmanship of the bearings ıall be of a relatively high order of accuracy.

The fact, otherwise inexplicable, that the conditions and laws f viscous lubrication were not discovered until the end of the ineteenth century, is doubtless due to the circumstance that it was nly at about that epoch that mechanical workmanship became enerally of such a quality that the necessary conditions were often omplied with. With rougher workmanship the necessary con-nuous films cannot be formed, but the two members of the bearing ome into actual or virtual contact, at least at some points, and thus ing about mixed conditions of solid and viscous friction incapable being referred to any simple or consistent laws

Even with workmanship which may be regarded as perfect the ltimate stage of failure initiated by any cause is contact of the solid ırfaces, either directly or through the small particles of solid im-urities which are always to some extent present in the lubricant here is thus suggested as a criterion of the safety of any bearing om such failure, the thickness of the lubricating film at its thinnest ıit under the working conditions which reduce this thickness to minimum

It will be seen from Table III, p 136, that for a bearing surface of ıven length, with given velocity, load, and lubricant, the thickness of ıe lubricant at the point of closest approach to the other surface, is a ıaximum when $\frac{a_2}{a_1} = 2{\cdot}2\ldots$ It is usual to adopt this ratio as that be preferred in designing bearings The table shows that with ıis ratio the coefficient of friction, though higher than is attainable ith greater values of the ratio $\frac{a_2}{a_1}$, is nevertheless already so small ıat its further reduction may usually be considered of little moment. must, however, be remembered that the optimum ratio, $\frac{a_2}{a_1} = 2{\cdot}2$ ften taken as 2 0 as a sufficiently close approximation) has, strictly eaking, been derived only from the special case of a bearing surface infinite length and for the condition $h = cx$.

It is hardly necessary to remark that if the velocity u_1 in the above ılculations be reversed, the equations for the pressures will be still

valid, with merely a change of sign for both u and p. It must, however, be remembered that whereas in the case of positive values of p the intensity of pressure has no necessary limit, negative values of p, that is to say tensions, are not in general sustainable in fluids such as ordinary oils, and indeed in most forms of bearings positive values of p less than Π, the atmospheric pressure, are usually inconsistent with the assumptions made in the calculations, since, under those conditions, air will be drawn into the spaces assumed to be occupied by oil.

The volume of fluid flowing between the surfaces per unit time may be calculated as follows:

From (30), p. 132, the rate of change of the pressure with x at the rear end of the bearing, i.e. at $x = a_1$, is

$$\frac{\partial p}{\partial x} = -\frac{6\mu u_1}{c^2}\left[\frac{1}{a_1^2} - \frac{2a_2}{a_1^2(a_1 + a_2)}\right]$$

$$= -\frac{6\mu u_1}{c^2 a_1^2}\left[\frac{a_1 - a_2}{a_1 + a_2}\right].$$

Therefore from (17), p. 122, and (25), p. 130, the volume rate at which the fluid passes through unit width of the normal plane at the rear end of the moving surface is

$$Q = -\frac{(ca_1)^3}{12\mu}\frac{6\mu u_1(a_1 - a_2)}{c^2(a_1 + a_2)a_1^2} + \frac{ca_1}{2}u_1$$

$$= \frac{ca_1 u_1}{2}\left(1 - \frac{a_1 - a_2}{a_1 + a_2}\right) = \frac{cu_1 a_1 a_2}{a_1 + a_2} \quad \ldots \quad (36)$$

The same result would be obtained by calculating the inflow at the front edge, and it may also be seen at once from the consideration that at the point of maximum pressure $x = x_1$, there being no flow due to rate of change of pressure, the volume rate at which fluid passes the normal plane is entirely due to the mean velocity $\frac{u_1}{2}$, acting over the film thickness, which is

$$cx_1 = \frac{2ca_1 a_2}{a_1 + a_2}.$$

Under the same assumptions as in Table III, p. 136, the value of Q for the condition $\frac{a_2}{a_1} = 2{\cdot}2$ is $2{\cdot}78 \times 10^{-4}$ c. c. per second per centimetre of the transverse dimension of the bearing.

Self-adjustment of the Positions of Bearing Surfaces

The question naturally arises how it is possible to secure in actual :arings the exact locations of the bearing parts shown to be necessary r the preceding calculations, and as illustrated in figs. 15–17, and ›w it is that so delicate an adjustment is not liable to be destroyed r inevitable wearing of the parts The explanation is that in suc- ssful types of bearings the parts are self-adjusting, their correct utual location being automatically brought about by their relative otion and continually corrected for any slight wear which may ke place.

Take for instance the case of the infinite plane slipper illustrated fig. 15, of which fig. 19 is a sec- ›n on any plane parallel to XZ.

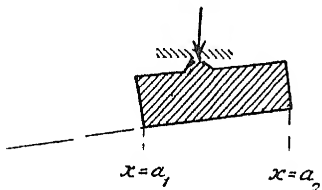


Fig 19

It has been seen from Table I that if the ratio of a_2 to a_1 is 2 the resultant pressure of the iid acts at the point $x = \bar{a}$, here $\bar{a} - a_1 = 0{\cdot}4221 \times (a_2 - a_1)$, id that with the value of h_1 ven in the table and unit values μ and u_1 the total resultant pressure is 1 Kgm per unit insverse width. Conversely, if a load of 1 Kgm per unit dth be applied to the slipper at the point $x = \bar{a}$ as indi- ted by the arrow in fig 19, and the slipper be moved with it velocity and supplied with fluid of viscosity 1 C.G S , it will ke up the same position. Experience, moreover, shows that such [uilibrium is stable for the displacements which are liable to occur the operation of the bearings. In the case of plane slippers the id must in practice be applied as shown in fig 19, that is to say, rough an actual or virtual pivot of some kind with which the slipper provided at the correct point. Actual examples will be illustrated the descriptions of thrust bearings given in the later parts of this apter.

In the case of cylindrical journal bearings, however, there is other mode of self-adjustment possible, which, though not so icient as the pivot method, is even simpler, and which undesignedly ok place in bearings of this class long before Reynolds' principle is discovered, and rendered them superior in efficiency to all other isses of bearings known at that time.

Self-adjustment in Journal Bearings

This action is illustrated for the ordinary form of fixed journal bearings in figs. 20*a*, *b*, *c*. We will assume that the bearing is one of a pair of journal bearings, as for the shaft of an electric motor, consisting of a cylindrical brass, or pair of semi-cylindrical half-brasses, of which only the lower half cylinder is normally effective. The radius of the bearing is, necessarily, greater than that of the journal. The load W is assumed to be the weight of the shaft and parts attached to it, acting vertically downwards

When the journal is at rest its position in the bearing is that

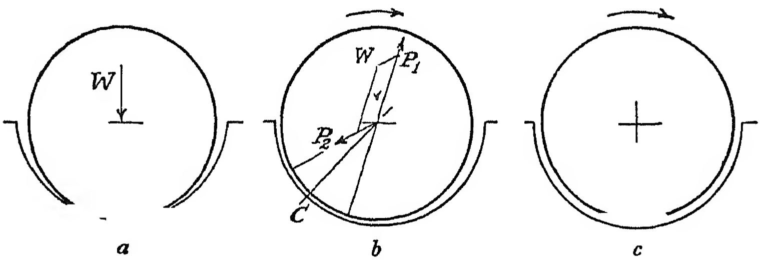


Fig 20

shown in fig 20*a*. The journal and bearing are then in contact along the lowest generating lines of their cylindrical surfaces When, however, the shaft begins to rotate, for example, in the clockwise direction as indicated in the figures, the oil at the right-hand side of the journal is subjected to a traction directed from the wider to the narrower part of the interspace between the journal and the bearing. On the principles which have been explained, the oil in this space will consequently exert a fluid pressure. On the opposite, or left-hand side of the journal, on the contrary, the interspace increases in thickness in the direction of motion, and consequently, as explained on p. 137, the pressure in the oil film will fall, becoming negative unless, as is sometimes the case, air is free to enter, when atmospheric pressure will tend to be established. The journal will consequently tend to move towards this left-hand side, the point of contact between journal and bearing shifting from the lowest generating lines to some higher line towards the left hand. Oil under pressure will thus be admitted between the parts of journal and bearing, and this action will be progressive until the resultant upward pressure becomes equal to the load W on the bearing. At

onstant speed a stable condition will be reached as shown in fig. 20*b*. 'he point of closest approach, C, will be somewhere on the left-hand de of the vertical, with a portion of the interspace above and to ıe left of C still diverging in the direction of motion The oil in ıis latter space will in consequence exert a negative pressure on ıe journal, as indicated by the arrow P_2 The resultant of this force nd the positive resultant pressure P_1, exerted by the oil in the ght-hand converging portion of the interspace, will be equal and pposite to W, the load on the journal.

If the speed of the journal is increased, the amount of con-ergence and divergence of the respective parts f the journal, for a given load W, will auto-ıatically diminish, the limiting condition with ıfinite speed (or zero load) being that illus-ated in fig 20*c*, the journal becoming then oncentric with the bearing

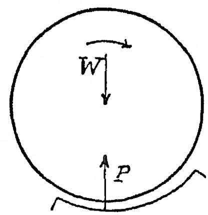


Fig 21

It may be noticed that in all cases the ıverging portion of the film, and the nearly arallel portions in the immediate neighbourhood f C, though of respectively negative and zero value for the support f load, are subject to shear of equal or greater intensity than the ffective pressure-producing film on the right hand and lower ırfaces of the journal For this reason such journal bearings, with ıe brasses embracing a semicircle or other relatively large arc, are ecidedly inefficient compared to a pivoted bearing of small arc such , that illustrated in fig 16, in which self-adjustment takes place ı the same mode as that described in connection with fig 19

It is also readily seen that, in all cases, the interspace between ıe journal and a segmental cylindrical bearing surface can only be onvergent throughout its length if the arc of the bearing surface less than 90° It is indeed desirable, in order to secure a fairly ıpid rate of convergence throughout, that the arc should be limited ı 45° at most

It will be seen that in such a case as that illustrated in fig 21, is possible, without pivoting the brass, for the resultant fluid ressure to be vertical and thus in equilibrium with the load W, ithout the formation of any diverging interspace, and this even hen the radius of curvature of the brass is the same as that ˋ the journal The latter is a convenient condition, as it lmits of the simplest and most accurate method of accurately ırming the bearing surface, namely by scraping or lapping it.

It is to be remarked, however, that automatic self-adjustment in journal bearings with rigid (i.e. non-pivoted) brasses, as in fig. 20*b* or 21, is only possible when there are not more than two bearings on a shaft, or if the shaft is flexible, as otherwise, since a third bearing will be invariably out of alignment to an extent comparable with or greater than the thickness of effective oil-films, it is not possible for each of the journals to adjust itself to its correct position. With any number of pivoted bearings, however, if the pivots are approximately in the vertical plane through the axis of the shaft, each bearing will exert a vertical resultant pressure, and by providing adjustments for the pivots in the vertical direction only it is possible to divide the total load carried by the shaft equally between the bearings.

Exact Calculation of Cylindrical Journal and Bearing

The mathematical solution of the viscous motion for the case illustrated in figs. 20*a*, 21 was given by Reynolds (5, p 158) The solution was simplified by Sommerfeld (8, p. 158), of whose process a brief résumé will now be given. In fig. 22, O and O′ are the centres and r and $r + \delta$ the radii of the cylindrical journal and semi-cylindrical bearing, both of infinite extension in the direction of their axes.

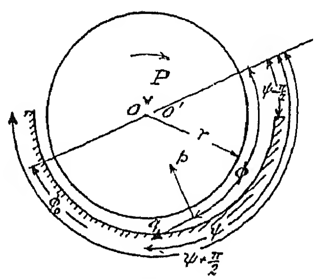


Fig 22

Let $OO' = \epsilon$, and $\frac{\delta}{c} = a$, a having therefore different values in different cases, varying from 1, when the journal and bearing are in contact, to ∞ when they are concentric.

Let ψ be the angle between OO′ and the vertical and ϕ an angular co-ordinate measured from the direction OO′, the co-ordinates for the ends of the bearing-brass being $\psi - \frac{\pi}{2}$, and $\psi + \frac{\pi}{2}$ as indicated in the figure. Thus the linear co-ordinate, x, in the direction of motion of the brass relatively to the journal is now *constant* $- r\phi$. Then from (27), p. 131,

$$\frac{dp}{rd\phi} = 6\mu u_1 \frac{h - h_1}{h^3}, \qquad (37)$$

and since $p = \Pi$ when $\phi = \psi - \frac{\pi}{2}$, and when $\phi = \psi + \frac{\pi}{2}$,

$$\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \frac{dp}{d\phi} d\phi = 6\mu u_1 r \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \frac{h - h_1}{h^3} d\phi = 0.$$

Also, if p is the fluid pressure and q the circumferential traction per unit width at ϕ, and P the total load on the bearing per unit width,

$$\mathrm{P}\cos\psi - \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} (p - \Pi)\cos\phi r d\phi - \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} q \sin\phi r d\phi = 0,$$

and

$$\mathrm{P}\sin\psi - \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} (p - \Pi)\sin\phi r d\phi + \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} q \cos\phi r d\phi = 0.$$

But since

$$\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} (p - \Pi)\cos\phi d\phi = \Big[(p - \Pi)\sin\phi\Big]_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} - \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \frac{dp}{d\phi}\sin\phi d\phi,$$

and

$$\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} (p - \Pi)\sin\phi d\phi = -\Big[(p - \Pi)\cos\phi\Big]_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} + \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \frac{dp}{d\phi}\cos\phi d\phi,$$

and since $p = \Pi$, both when $\phi = \psi + \frac{\pi}{2}$ and when $\phi = \psi - \frac{\pi}{2}$, so that the terms not under integral signs vanish,

$$\left.\begin{aligned} &\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \left(\frac{dp}{d\phi} - q\right)\sin\phi d\phi = -\frac{\mathrm{P}}{r}\cos\psi, \\ \text{and} \quad &\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \left(\frac{dp}{d\phi} - q\right)\cos\phi d\phi = +\frac{\mathrm{P}}{r}\sin\psi \end{aligned}\right\} \quad \ldots\ldots.(38)$$

Now from (33), p. 133, and (37), p. 142,

$$\frac{1}{r}\left(\frac{dp}{d\phi} - q\right) = 6\mu u_1 \frac{h - h_1}{h^3} + \frac{\mu u_1}{rh},$$

in which the second term on the right can be neglected on account of the smallness of h compared to r.

Thus the equations can be written

$$\left.\begin{aligned} 6\mu u_1 r\int_{\psi-\pi/2}^{\psi+\pi/2}\{(h-h_1)/h^3\}\sin\phi\, d\phi &= -(\mathrm{P}/r)\cos\psi, \\ 6\mu u_1 r\int_{\psi-\pi/2}^{\psi+\pi/2}\{(h-h_1)/h^3\}\cos\phi\, d\phi &= (\mathrm{P}/r)\sin\psi, \end{aligned}\right\} \quad . . (39)$$

or, since $h = \epsilon(\alpha + \cos\phi)$, $h_1 = \epsilon(\alpha + \cos\phi_1)$,

these equations become

$$\left.\begin{aligned} \int\frac{\sin\phi}{(\alpha+\cos\phi)^2}d\phi - \frac{h_1}{\epsilon}\int\frac{\sin\phi}{(\alpha+\cos\phi)^3}d\phi &= -\frac{\delta^2\mathrm{P}\cos\psi}{6\mu a^2 r^2 u_1}, \\ \int\frac{\cos\phi}{(\alpha+\cos\phi)^2}d\phi + \frac{h_1}{\epsilon}\int\frac{\cos\phi}{(\alpha+\cos\phi)^3}d\phi &= \frac{\delta^2\mathrm{P}\sin\psi}{6\mu a^2 r^2 u_1}, \end{aligned}\right\} \quad . (40)$$

the integrals as before being from $\psi - \frac{\pi}{2}$ to $\psi + \frac{\pi}{2}$.

These integrations can be effected by usual methods,* and from the results Sommerfeld calculated the following numerical table, Table IV, in which $\eta_0 = \frac{\delta}{r}$, and the "coefficient of friction", $f = \frac{\mathrm{M}}{\mathrm{P}r}$, where M is the moment of the frictional tractions about O.

TABLE IV

α^2	ψ	$\cos\phi_0$	u_1	f
1	90°	− 1·0	0	$\eta_0 \times$ 1 00
1·02	120°	− 0 998	$\frac{\mathrm{P}\eta_0^2}{\mu} \times$ 0 012	× 0 94
1·13	129°	− 0 98	× 0·04	× 0 93
1·5	135°	− 0 93	× 0 08	× 0 92
2·4	133°	− 0 88	× 0·14	× 1 00
6 3	128°	− 0 72	× 0·29	× 1 34
33·9	120°	− 0 50	× 0·62	× 2 17
∞	90°	0	× ∞	× ∞

If the coefficient of friction f be plotted with u_1 as the variable,

* $$\int\frac{\cos\phi}{\alpha+\cos\phi}d\phi = \int\frac{(\alpha+\cos\phi)-\alpha}{\alpha+\cos\phi}d\phi = \phi - \alpha\int\frac{d\phi}{\alpha+\cos\phi}$$
$$= \phi - \alpha\frac{2}{\sqrt{\alpha^2-1}}\tan^{-1}\left(\tan\frac{\phi}{2}\sqrt{\frac{\alpha-1}{\alpha+1}}\right)$$

Differentiate this with respect to α to get integrals in second line of (40). Integrals in first line come at once, since $d(\alpha + \cos\phi) = -\sin\phi d\phi$.

P being constant, then for various values of $\eta_0 = \frac{\delta}{r}$ we have a series of curves, in which as u_1 increases from zero, the coefficient of friction falls at first to a minimum value about 8 per cent lower than its initial value, then with further increase of u_1 the coefficient gradually rises and finally increases to an asymptotic approximation to the straight line $f = \frac{\mu\pi r u_1}{P\delta}$.

The value of u_1 for which the coefficient of friction is a minimum is approximately

$$u_0 = \frac{P\delta^2}{12\cdot 5 \times \mu r^2}.$$

In actual bearings the initial value of the coefficient of friction will be much higher than that calculated, since with very low velocities, and values of a only slightly greater than 1, the journal and bearing will be, owing to minute roughnesses of their surfaces, in metallic contact instead of being separated by a very thin continuous film, as assumed in the theory

It is to be observed that in these calculations of Sommerfeld's the portion of the film between the point of closest approach, $\phi = \phi_0$, and $\phi = \psi + \frac{\pi}{2}$ is subject to a negative pressure. The possibility of such a condition may reasonably be postulated in very wide bearings, but can hardly be assumed in bearings of usual proportions unless special means are employed for preventing the entry of air at the sides

Approximate Calculation of Cylindrical Bearings

The method and results of Sommerfeld's investigation given above apply to the case of a cylindrical bearing whose angular length in the circumferential direction is 180°. A similar process may be applied to bearings of smaller angular length, as in fig 21, such bearings, as explained on p 141, being preferable in practice In these cases, however, there is little value in the assumption that the surface of the brass is a circular cylinder, and, especially in the pivoted type, it is usually sufficient to assume that the thickness of the interspace is a linear function of ϕ, that is to say to apply in the case of a very wide bearing the method and results of pp. 131-133.

If a closer approximation is desired, the form of the bearing may

be approximately represented by the equation $h = cx^m = C(r\phi)^m$ with an appropriate value of m differing from unity The solution of this case has been given by Rayleigh (20, p. 159), who, however, found that in the numerical applications which he made of it, the results did not differ very materially from those derived from the simpler formula, $h = cx$.

Plane Bearings of Finite Width

A more important modification of the Reynolds' theory of bearings with uniformly varying interspaces, is that which it requires for its application to bearings which are of limited width, and in which, consequently, there is a transverse flow of the fluid under pressure to the sides, i e in the direction of Y, as well as flow in the direction of the relative motion X.

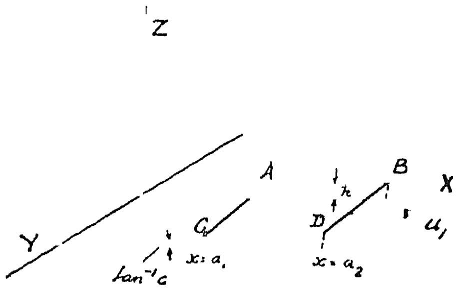


Fig 23

The solution, as given by Michell (9, p 158), involves rather lengthy calculations, and we can give only an indication of the method and a few working formulæ and constants

In fig 23 (which corresponds to fig 15 for the case of infinite width), ABDC is a rectangular plate in the plane $z = cx$ (the length of the plate being $a_2 - a_1$, and its width b) sliding in the direction of X with velocity u_1

The pressure is assumed to be uniform everywhere except in the interspace between the plate ABDC and the infinite fixed plate in the plane $z = 0$, i e the boundary conditions of the plate are $p = \Pi$, when $x = a_1$, or $x = a_2$, for all values of y, and also when $y = 0$, or $y = b$, for all values of x Between the two plates p must satisfy the differential equation (26), p 130, i.e

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right) + 6\mu\left(u_1\frac{\partial h}{\partial x} + v_1\frac{\partial h}{\partial y}\right) = 0;$$

$$\text{or, since} \quad h = cx, \text{ and } \frac{\partial h}{\partial y} = 0,$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{3}{x}\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial y^2} + \frac{6\mu u_1}{c^2 x^3} = 0. \ldots\ldots\ldots(41)$$

This equation may be written in the form

$$\frac{\partial^2 p}{\partial x^2} + \frac{3}{x}\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial y^2} + \frac{6\mu u_1}{c^2 x^3}\frac{4}{\pi} \times$$
$$\left(\sin\frac{\pi y}{b} + \frac{1}{3}\sin\frac{3\pi y}{b} + \ldots + \frac{1}{m}\sin\frac{m\pi y}{b} + \ldots\right) = 0,$$

since the sum of the series in brackets, for all the values of y with which we are concerned, viz. $y = 0$ to $y = b$, is $\frac{\pi}{4}$.

To solve this differential equation so as to give p as a function of x and y, it is assumed that there is a solution of the form

$$p = \Pi + p_1 + p_3 + \ldots + p_m + \ldots \text{ ad inf.} \quad \ldots\ldots(42)$$

$$\text{in which} \quad p_m = \frac{\xi_m \sin\frac{m\pi y}{b}}{\frac{m\pi x}{b}},$$

ξ_m being a function of x only.

The integer m can have only odd values, because $p - \Pi$ must be symmetrical on both sides of $y = \frac{b}{2}$.

$$\text{Thus} \quad p - \Pi = \Sigma_1^\infty \frac{\xi_m \sin\frac{m\pi y}{b}}{\frac{m\pi x}{b}},$$

where m is odd.

If for brevity we write $24\frac{\mu u_1}{bc^2} = k$, and $\frac{m\pi x}{b} = \zeta$,

$$\frac{\partial p}{\partial x} = \frac{m\pi}{b}\frac{\partial p}{\partial \zeta} = \frac{m\pi}{b}\Sigma\left\{\frac{1}{\zeta}\frac{\partial \xi_m}{\partial \zeta} - \frac{\xi_m}{\zeta^2}\right\}\sin\frac{m\pi y}{b},$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{m^2\pi^2}{b^2}\frac{\partial^2 p}{\partial \zeta^2} = \frac{m^2\pi^2}{b^2}\Sigma\left\{\frac{1}{\zeta}\frac{\partial^2 \xi_m}{\partial \zeta^2} - \frac{2}{\zeta^2}\frac{\partial \xi_m}{\partial \zeta} + \frac{2\xi^m}{\zeta^3}\right\}\sin\frac{m\pi y}{b},$$

$$\frac{\partial^2 p}{\partial y^2} = -\Sigma\frac{m^2\pi^2\xi_m}{b^2\zeta}\sin\frac{m\pi y}{b}.$$

Thus the coefficient of $\sin\frac{m\pi y}{b}$ in equation (41), p. 146, is

$$\frac{m^2\pi^2}{b^2\zeta}\left\{\frac{\partial^2 \xi_m}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial \xi_m}{\partial \zeta} - \left(1 + \frac{1}{\zeta^2}\right)\xi_m - \frac{k}{\zeta^2}\right\} = 0. \quad \ldots\ldots(43)$$

Every such coefficient must vanish, and consequently the factor within the brackets may be equated to zero, of which equation the particular integrals are the Bessel's Functions, $I_1(\zeta)$ and $K_1(\zeta)$, and the complete integral may be written in either of the forms

$$\xi_m = A_m I_1(\zeta) + B_m K_1(\zeta) - k(1 + \frac{\zeta^2}{3} + \frac{\zeta^4}{5\ 3^2} + \ldots), \quad \ldots .(44)$$

$$\text{or} \quad \xi_m = A'_m I_1(\zeta) + B'_m K_1(\zeta) - k(\zeta^{-2} + 3\zeta^{-4} + 5.3^2\zeta^{-6} \ldots) \ldots (45)$$

The second form, useful when ζ is very large, being "asymptotic"

The coefficients A_m, A'_m, B_m, B'_m are to be determined so as to make ξ_m vanish for $x = a_1$, and $x = a_2$, and hence p_m vanish for all values of y on these two lines

These coefficients can only be determined arithmetically, numerical values being given to the quantities a_1, a_2, and b. The steps of the calculation, with tables, are given in the paper (9, p 158)

The coefficients A_m, &c , having been calculated, the values of p for as many points x, y, as may be desired are also calculated arithmetically, and when p is known the total fluid pressure supporting the block is determined by arithmetical or graphical summation, from the relation

$$P = \int_{a_1}^{a_2}\int_0^b p dx dy. \quad \ldots \quad (46)$$

The frictional traction by (33a), p 133, is $F = \frac{\mu u_1}{c} \log_e \frac{a_2}{a_1}$, per unit width, and

$$Fb = \frac{\mu u_1}{c} b \log_e \frac{a_2}{a_1} = \frac{\mu u_1}{c}\sqrt{A} \log_e \frac{a_2}{a_1} \quad .(47)$$

for the whole of the square shoe, of area $A = b(a_2 - a_1)$.

The point of action of the resultant pressure is found by an arithmetical summation of moments By way of examples, a few numerical formulæ will be given.

The total pressure on a square bearing in which

$$a_2 - a_1 = a_1 = b \text{ is } P = \frac{0{\cdot}0669 \mu u_1 A^{\frac{3}{2}}}{c^2},$$

being, by comparison with formula (31), p. 132, only 0·421 of the total pressure on a portion of equal area, equal length and inclination of a plane of infinite width, thus showing the effect of the escape of oil from the sides of the bearing.

The position of the centre of pressure for the finite square block is at a distance $0{\cdot}42a_1$ from the rear edge, as compared with $0{\cdot}431a_1$, in the infinite bearing. (See Table III, p. 136.)

The coefficient of friction $\dfrac{F}{P}$ is $10{\cdot}3c$. A further calculation serves to show that of the total quantity of oil which enters the interspace at the leading edge of the square shoe approximately one-sixth passes out at each of the sides and the remaining two-thirds at the rear edge.

Similarly, in the case of a bearing whose width transverse to the motion is only one-third of its length, so that

$$a_2 - a_1 = a_1 = 3b,$$

the total pressure is

$$P = \frac{0{\cdot}0146\mu u_1 a_1}{c^2} = \frac{0{\cdot}0253\mu u_1 A^{\frac{1}{2}}}{c^2},$$

and the centre of pressure is $0{\cdot}39a_1$, from the rear end.

These results, as already explained, are equally applicable to journal bearings as to plane slide bearings provided that the form of the bearing surface and the position of the pivot are such that $h = cx$, and $a_2 - a_1 = a_1$.

Arithmetical evaluations of the pressures and frictional coefficients given by the above theory have been calculated for an extensive series of bearing blocks of varying proportions by Torao Kobayashi (30).

Cylindrical Bearings of Finite Width

Mathematical treatment of cylindrical bearings of finite width, corresponding to the theory given above for plane bearings, does not yet exist. This is unfortunate, since the limitation of width has an even greater effect in a cylindrical than in a plane bearing in reducing the pressures generated, and particularly so if the arc subtended by the cylindrical bearing approaches a semicircle as it usually does in the conventional type of journal bearing.

By way of illustration of this statement, we may take a journal bearing in which the bearing-shell subtends an arc of 120°, and in which the thickness of the film at the inlet is double its thickness at the outlet (as in the plane bearing previously discussed).

Such a bearing is illustrated in fig. 24,* in which r is the radius

* From (28), p. 159.

of the journal, $r + \delta$ that of the bearing-shell, the distance between their centres being ϵ, while $2b$ is the width of the bearing.

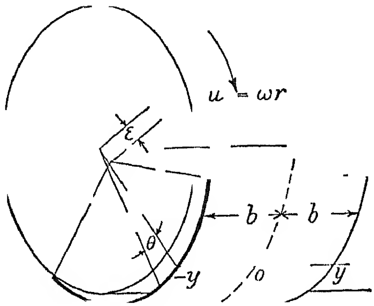


Fig 24

It is readily seen that in such a bearing the areas through which oil may escape at the sides of the bearing are much greater relatively to the areas at the front and rear (at which the oil would enter and leave in a two-dimensional bearing) than is the case in a plane bearing of similar length and width.

In the particular case in which the width of the bearing is equal to the radius of the journal, and in which the radii of journal and bearing are equal, so that

$$\delta = 0$$
$$2b = r,$$

the respective areas of the leading, trailing and side openings are in the proportions

$$2 : 1 : 10{\cdot}3.$$

In other words, the oil which enters the interspace at the leading edge has more than 10 times greater area by which to leave at the sides than at the rear Pressures are consequently determined almost entirely by the conditions at the sides, and a two-dimensional solution would convey a very false idea of the actual conditions

A more useful approximation in such a case can be obtained by treating the bearing as infinitely long in comparison with its width. On this assumption the pressures over the portion of the bearing near its trailing end (which is the only portion in which effective pressures will be generated) are given by

$$p = \frac{3\mu\omega\epsilon \sin\theta}{h^3}(b^2 - y^2),$$

being the axial co-ordinate measured from the middle circumference, and h the varying thickness, determined by

$$h = \delta + \epsilon \cos\theta.$$

Experimental Results

The curves given in the right-hand half of fig 24*a* are derived rom an extensive series of tests of a pivoted journal bearing of which he circumferential length was 6 98 cm. and the width was 6·35 m, the block being thus not quite square From examination of his diagram it will be found that for a given load the coefficient of riction varies approximately as $\sqrt{\mu u_1}$, while for a given value of μu_1, varies nearly inversely as the square root of the load, both these esults being in accordance with the formulæ above The facts tated are brought out more explicitly in the following table, which hows that the values of F, P and μu_1, as read off the right-hand art of fig 24*a*, make $F\sqrt{(P/\mu u_1)}$ approximately constant. The eft-hand part of fig 24*a* will be explained on p. 157.

TABLE V

F	$P/\mu u_1$	$\sqrt{(P/\mu u_1)}$	$10^3 F\sqrt{(P/\mu u_1)}$
0 0008	0 12	0 35	0 28
0012	067	26	31
0016	037	19	30
0020	023	15	30
·0024	017	·13	31

Types of Pivoted Bearings

The chief practical field of application of *plane* pivoted bearings to thrust bearings These usually take the form of an annular eries of " shoes " or " blocks " pivoted upon fixed points in the ationary casing and presenting their plane working surfaces to a lane-surfaced annular collar fixed on the rotating shaft. Such a

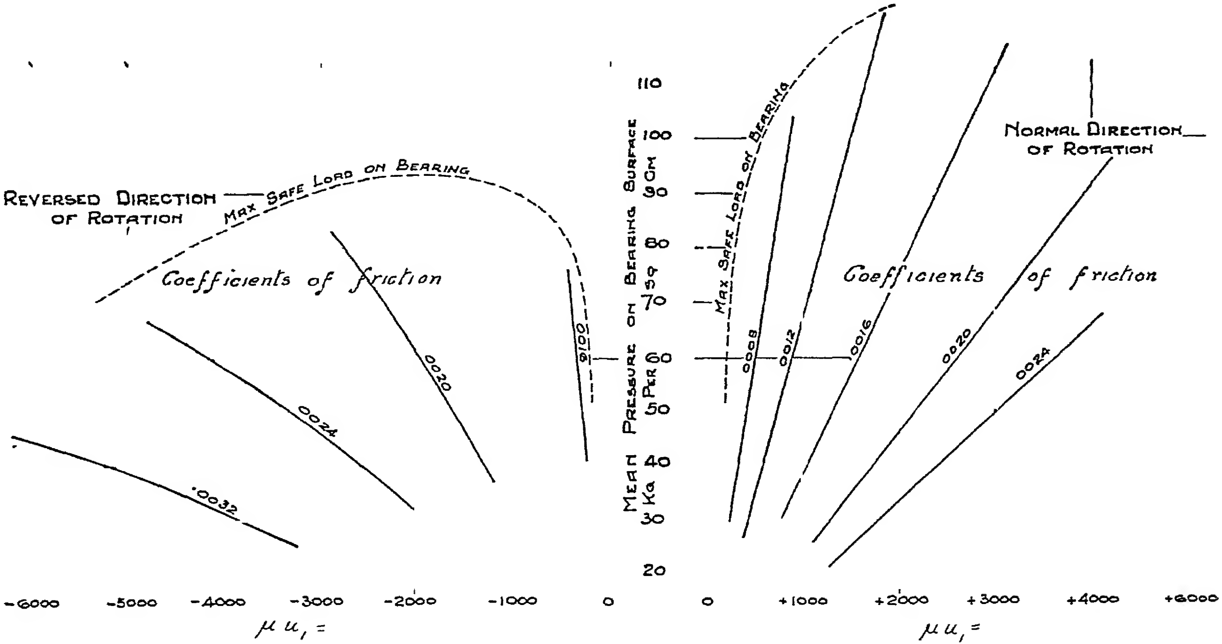


Fig 24a —The base is the product of the viscosity, μ of the oil at air temperature, and the surface speed of the journal (positive or negative) both in C G S units

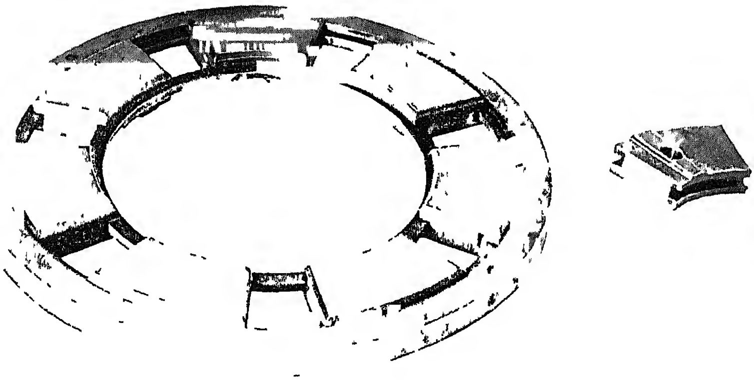


FIG. 26.—THRUST SHOES

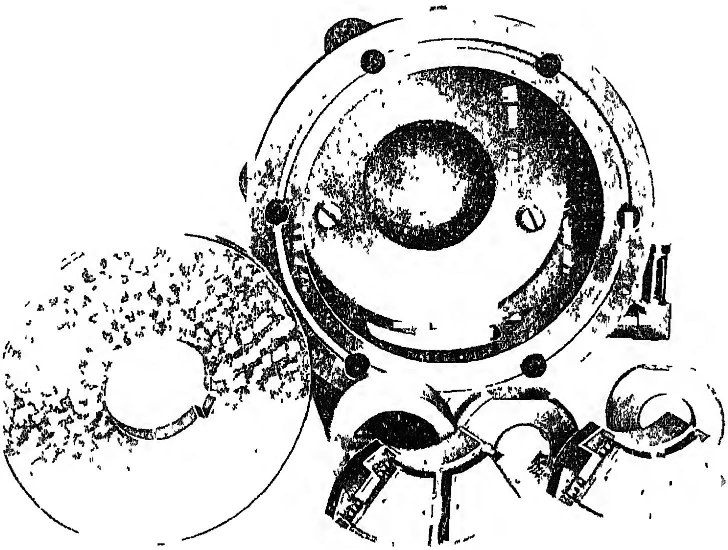


FIG. 28.—THRUST BEARING FOR HORIZONTAL SHAFTS.

Facing page 15-

rust bearing arranged for a vertical shaft is shown in fig. 25. In is bearing the thrust shoes, t, are fixed in the lower part of the sing of the bearing which also serves as the casing of a journal earing for the thrust shaft. (The journal bearing is of the flexibly-voted type described on p. 155.)

In order that the casing may form a reservoir for oil to

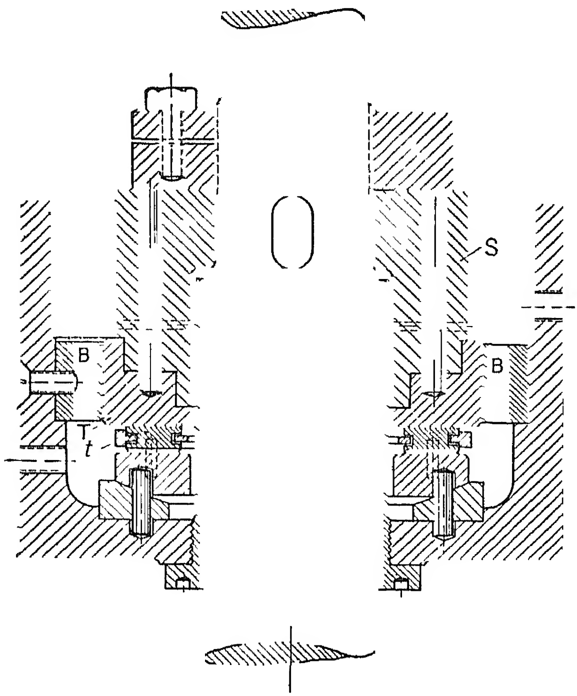


Fig 25

ubricate both bearings, the journal surface is formed not directly pon the shaft, but on the outer surface of a collar T, attached to e sleeve S, and forming also the thrust collar which revolves upon e annulus of thrust shoes t. This annulus is shown separately in g 26 (The flexible journal ring is shown in fig. 31, facing p. 154).

In figs 27 and 28 is shown another type of thrust bearing, con-enient for application to horizontal shafts. In this form, which adapted to take thrusts in either axial direction, two pivoted thrust-hoes R only are employed for each direction of thrust, each pair eing mounted in a common housing H, which is itself pivoted on the

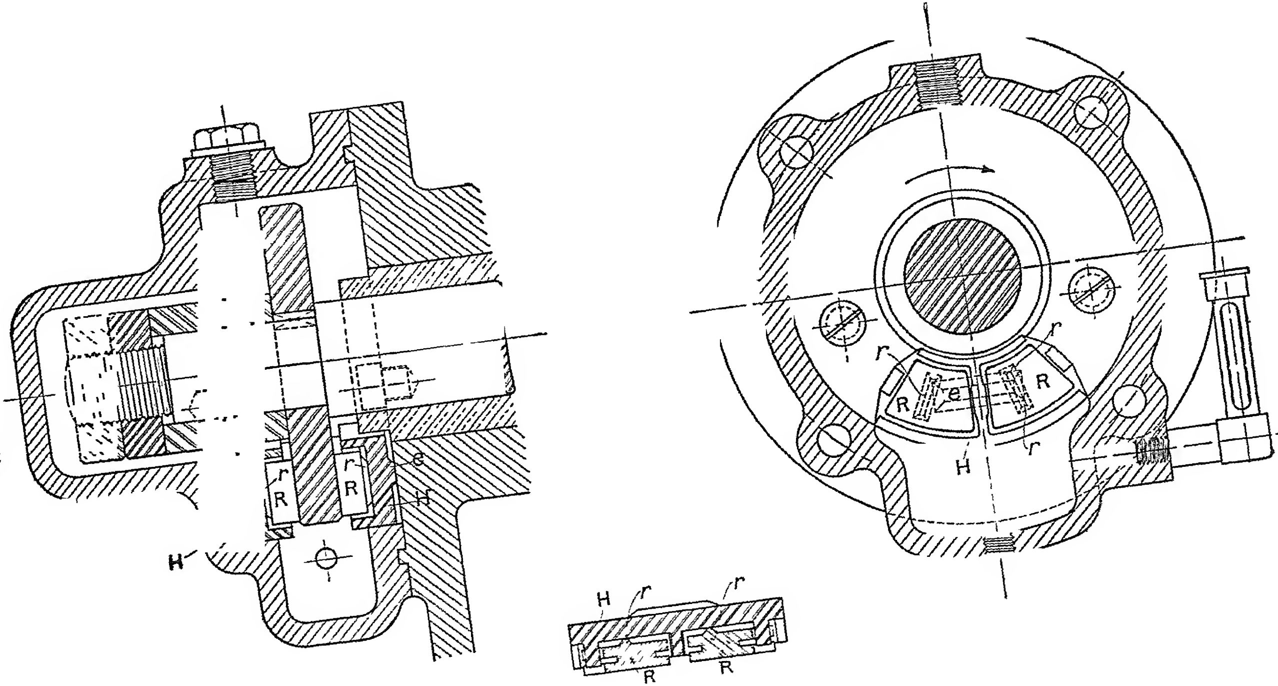


Fig 27

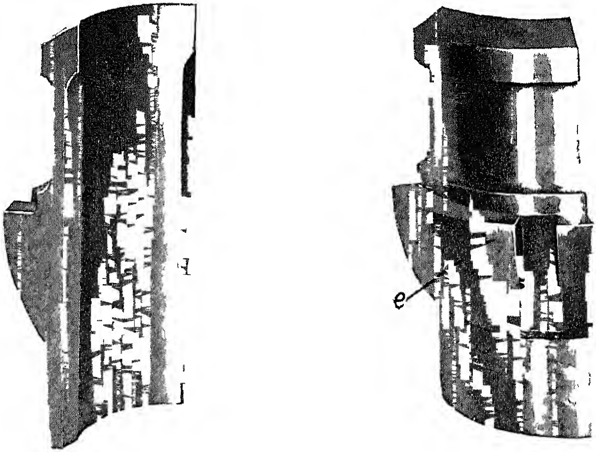


FIG. 29 VIEWS OF PIVOTED JOURNAL BEARING SHOE

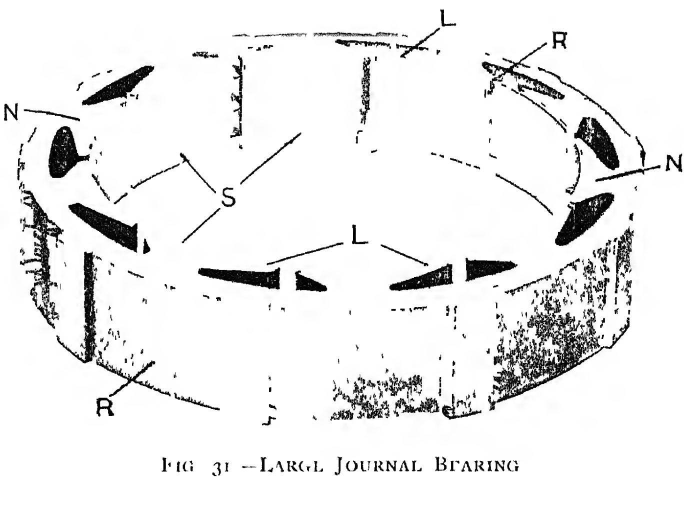


FIG 31 —LARGE JOURNAL BEARING

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Indian Institute of Science
Bangalore 3

ower part of the fixed casing on an edge *e* at right angles to the ivoting ribs, *rr*, of the individual shoes.

Fig 28 is a photograph of the parts of the bearing which is shown y longitudinal and cross sections in fig. 27.

In fig. 29 are shown two views of a pivoted journal bearing shoe, eing one of an annular series of four arranged for the journal bearing f a vertical shaft The pivoting edge *e* is clearly seen on the back f the shoe

Flexible Bearings

The comparatively small clearances and slight relative inclinations etween coacting bearing parts, requisite to produce effective lubriating films, allow of a modified type of contruction for achieving the same purposes as re attained by pivoted bearings It is evident hat in these a spring, or other continuous but leformable connection, may be substituted for he rolling or rotating contact of a pivot. Such spring may be either a separate part attached oth to the shoe and its supporting member, or nay be an integral part of one or both of these provided such part s made with the necessary degree of flexibility to allow of the shoe leflecting under the load. Alternatively, as in a type of construction roposed by Ferranti,* a pair of springs may be used to connect the hoe and its support, viz a comparatively stiff spring at the rear and lighter spring at the front of the shoe This construction, which is llustrated in fig 30, will evidently have the effect of applying he resultant load at a point P, behind the middle point of the hoe, much as if it were applied to a rigid pivot at that point.

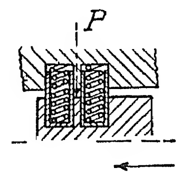

Fig 30

The chief advantage of a flexible construction is that it enables mall or relatively unimportant bearings to be simplified, by constructing a number of bearing shoes integral with, but flexibly connected to, a common supporting member. A serious disadvantage s that the flexibility involves more or less risk of fracture of the lexible part, a danger which is to some extent overcome by giving lexibility to a portion of the shoe itself. The large journal bearing, already mentioned on p. 153, is constructed in this way, and is llustrated in fig. 31, and in fig. 25, p. 153.

In the former the individual shoes, S, may be seen attached to the

* British patent No. 5035/1910.

supporting ring R, by flexible necks N, and having also their leading portions, L, reduced in thickness for some distance from the leading edges.

Limitations of the Theory

As the shoes of such thrust bearings as are illustrated in figs 24*a* to 28 are usually of small radial width compared to their mean radii, the formulæ given for rectangular bearing slippers may usually be applied to them with sufficient accuracy for practical purposes in spite of their sectorial form. A more exact calculation can be made when required by a process which refers the co-ordinates of the sectorial shoe to those of the rectangular shoe.*

Of greater practical importance are the departures from the results of the calculations which in some cases arise from the insufficiency of the physical assumptions which have been made, especially as to the constancy of the coefficient of viscosity

An experimental method of solution, imagined and applied by Kingsbury (29), is free from most, if not all, of these limitations This method utilizes the identity which exists between the equations connecting pressure and volume-flow in viscous liquids, and potential and current in an electrical conductor The conductor used is a conducting liquid contained within solid, non-conducting boundaries shaped to represent in correct proportion (though on an exaggerated scale as regards thickness of the conductor) the lubricating film to be investigated The results obtained by Professor Kingsbury agree closely with those of the mathematical investigations, e g those of the plane bearing of finite width given on pp. 146 of the present chapter. The method has been applied to both plane and cylindrical bearings of various ratios of length to width

It was shown in Table I, p. 118, that the viscosity of lubricating oils diminishes rapidly as the temperature rises. In a well-loaded pivoted bearing, carrying for instance a mean pressure of 70 Kgm per square centimetre, and with the product μu_1 amounting to 2000 C G S., and with usual dimensions, it can easily be deduced from calculations of the energy expended in overcoming the viscous friction, and of the heat capacity of the quantity of oil flowing through the lubricating film, that apart from conduction of heat through the metal, the oil would rise in temperature some 50° C. in passing

* See (16, p. 158) Correspondence.

rough the bearing. Conduction will diminish this rise of tempera-ture, but in most cases of heavily loaded bearings it is still sufficient make the viscosity of the oil in the rear portion of the film much wer than in the leading portion. Thus, other conditions remaining naltered, the outflow of oil at the rear will take place with a less pid fall of pressure in that direction, and the point of maximum essure will be shifted towards the front of the bearing. In fig. 18 le dotted curve $p\frac{1}{2}$, p. 135, *Revue B.B.C.* (19, p. 159), is figured the assumption that the rise of temperature of the oil is such at its viscosity at exit is reduced to one-half of its value at entry, le conditions being otherwise the same as those for the full-line irve as already explained on p. 135.

The lower values of the fluid pressure throughout the film and e shift of the point of maximum pressure towards the leading edge e clearly seen. The point of action of the resultant pressure is so moved forward relatively to its position with constant oil tem-rature, and it may even happen that the centre of pressure is at, in front of, the middle point of the bearing block If, for example, e direction of motion of a pivoted bearing is reversed, so that the vot is before instead of behind the centre of the bearing, it is ll possible in many cases for a lubricating film to be formed and essures generated in it in equilibrium with the load Such an ect is shown in the left-hand half of fig 24*a*, which shows the sults of reversing the bearing In such a case the oil film is neces-rily thinner, and the coefficient of friction higher than for the rrect direction of motion, but nevertheless the capability of being versed in this manner, and of even then working with coefficients of ction lower than those of non-pivoted bearings, is a valuable operty of the pivoted type When, however, pivoted bearings are nployed in this manner, it has always to be remembered that their ccess when running reversed depends upon the lubricant having a nsiderable rate of diminution of viscosity with rising temperature or example, an experimental thrust bearing which ran very suc-ssfully in both directions with water and with a mineral oil of low scosity as lubricants, or with carbon bisulphide when running in e normal direction, completely failed to run in the reversed direc-n with the last-named fluid, doubtless on account of the peculiarity its viscosity-temperature relation, which has already been men-ned on p 117

Effects of the same nature, which arise in the use of air as a oricant in pivoted thrust bearings, have been pointed out and

experimentally investigated by Stone * (24, p. 159). With air, owing to the viscosity of gases increasing with rising temperatures instead of diminishing as in liquids, pivoted bearings tend to be much less stable as to the inclination of the pivoted shoe than with liquid lubricants.

On the other hand, as the same author has also remarked, the increase of the viscosity of the air film with temperature tends to increase the thickness of the film when a rise of temperature takes place owing to excessive load or undue resistance. The risk of direct contact of the bearing elements thus tends to become less as the bearing heats up, instead of greater as with liquid lubricants.

Calculation and experiment agree in showing that the successful use of air as a lubricant demands the highest refinements of workmanship, with moderate loads and relatively high speeds.

BIBLIOGRAPHY OF ORIGINAL WORKS ON VISCOSITY OF FLUIDS AND VISCOUS THEORY OF LUBRICATION

1 POISEUILLE. "Recherches expérimentales sur le mouvement des liquides" *Mémoires de l'Académie des Sciences*, **9**, 1846
2. STOKES, G G "Theories of the Internal Friction of Fluids in Motion, etc" *Collected Papers*, Vol. I, p. 75
3. HIRN, A "Études sur les principaux phénomènes que présentent les frottements médiats, etc" *Bulletin de la Société Industrielle de Mulhouse*, 1855
4 BEAUCHAMP TOWER *Proc. Inst Mech Eng*, 1883 and 1884
5 OSBORNE REYNOLDS "On the Theory of Lubrication" *Phil Trans Roy Soc London*, 1886, p. 157, also *Collected Papers*, Vol II, p 228
6 GOODMAN *Manchester Association of Engineers*, 1890
7. LASCHE "Die Reibungsverhaltnisse in Lagern" *Zeitschrift deutscher Ingenieure*, 1902
8. SOMMERFELD. "Hydrodynamische Theorie der Schmiermittelreibung." *Zeitschrift fur Math u Phys.*, 1904, **50**, p 97.
9 MICHELL, A. G M. "The Lubrication of Plane Surfaces" *Zeitschrift fur Math u. Phys*, 1905, **52**, p 123
10 BRILLOUIN *La Viscosité* (Gauthier-Villars, Paris, 1907)
11 HOSKING "Viscosity of Water" *Phil Mag*, April, 1909, p 502

* These experiments were made by means of a thrust bearing consisting of quartz-crystal thrust shoes and a glass thrust collar, the bearing surfaces being worked to true planes by optical methods Monochromatic diffraction bands produced by the closely adjacent pair of bearing surfaces at a slight mutual inclination gave an immediate and very accurate measure of the thickness of the lubricating film

2 ARCHBUTT and DEELEY. *Lubrication and Lubricants*, 2nd ed. (London, 1912)

3. CAROTHERS, S P. "Portland Experiments on the Flow of Oil in Tubes." *Proc. Roy Soc*, A, **87**, No. A. 594, Aug, 1912

4. FAUST, O. "Internal Friction of Liquids under High Pressure." *Gottingen Institute of Phys. Chem.*, 21st June, 1913. (Quoted in *Report of British Lubricants and Lubrication Inquiry Committee*, 1920)

5 GUMBEL "Das Problem der Lagerreibung" *Berliner Bezirks-verein deutscher Ingenieure*, 1st April, 1914

6 NEWBIGIN, H T. "The Problem of the Thrust Bearing" *Min. Proc Inst. C E*, 1914

7 MARTIN, H M "Theory of Lubrication" *Engineering*, July, 1915, p 101.

8 STONE, W "Viscometer" *Engineering*, 26th Nov, 1915

9. "De F" "Paliers de butée modernes" *Revue B B C*, Jan–April, 1917

0 RAYLEIGH, LORD "On the Theory of Lubrication" *Collected Papers*, Vol VI, p 523

1 HYDE, J H. "Viscosities of Liquids (Oils) at High Pressures" *Proc Roy Soc A*, **97**, No A 684, May, 1920.

2 MARTIN, H M "The Theory of the Michell Thrust Bearing" *Engineering*, 20th Feb, 1920

3 LANCHESTER, F W "Spur Gear Erosion" *Engineering*, 17th June, 1921, p 733

1 STONE, W "A Proposed Method for Solving Problems in Lubrication" *The Commonwealth Engineer* (Melbourne), Nov, 1921

5 STONEY, G "Journal Bearings" *Engineering*, 3rd March, 1922

6 HERSEY, M. D and SHORE, H "Viscosity of Lubricants under Pressure" *Amer Soc of Mech Engineers*, Dec, 1927

7 BOSWALL, R O "The Theory of Film Lubrication", pp xi, 280 (Longmans, London & New York, 1928)

3. MICHELL, A G M "Progress of Fluid-film Lubrication" *Trans of Amer Soc of Mech Engineers*, M S P 51/21, Sep–Dec, 1929

) KINGSBURY, A "On Problems in the Theory of Fluid-film Lubrication, with an experimental method of solution" *Amer Soc of Mech Engineers*, Dec, 1930

) KOBAYASHI, TORAO "A Development of Michell's Theory of Lubrication" *Report of the Aeronautical Research Institute*, Tokyo University, No 107, June, 1934

A comprehensive bibliography of the whole subject is contained in . Notes on the History of Lubrication", Parts I and II, by M D Hersey, *urnal Amer. Soc of Naval Engineers*, Nov, 1933 and Aug., 1934.

CHAPTER IV

Stream-line and Turbulent Flow

Stream-line Motion

The motion of a fluid may be conveniently studied by considering the distribution and history of the *stream lines*, i.e the actual paths of the particles. If these paths or stream lines preserve their configuration unchanged, the motion is called *steady* or *stream-line* motion. (See Chapter II, p. 57.)

If the stream line be imagined to form the axis of a tube of finite sectional area having imaginary boundaries, and such that its area at different points in its length is inversely proportional to the velocities at these points, this is termed a "stream tube".*

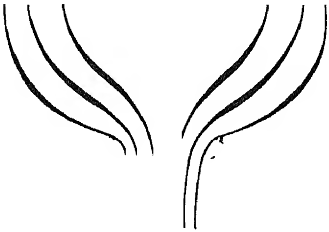

Fig 1

Such stream lines must always have a continuous curvature, since, to cause a sudden change in direction, an infinite force acting at right angles to the direction of flow would be necessary. It follows that in steady motion a fluid will always move in a curve around any sharp corner, and that the stream lines will always be tangential to such boundaries, as indicated in fig. 1, which shows the general form of the stream lines of flow from a sharp-edged orifice. With a very viscous fluid, the effect of cohesion may introduce comparatively large forces, and the radius or curvature may then become very small.

* See an alternative way of putting this idea, Chapter II, p. 57.

Stability of Stream-line Motion

Several conditions combine to determine whether, in any partiular case of flow, the motion of a fluid shall be stream-line or turulent. Osborne Reynolds, who first investigated the two manners [motion by the method of colour bands,* came to the conclusion iat the conditions tending to the maintenance of stream-line motion e

(1) an increase in the viscosity of the fluid;
(2) converging solid boundaries,
(3) free (exposed to air) surfaces;
(4) curvature of the path with the greatest velocity at the outside the curve;
(5) a reduced density of the fluid.

he reverse of these conditions tends to give rise to turbulence, does a state of affairs in which a stream of fluid is projected into body of fluid at rest.

The effect of solid boundaries in producing turbulence would pear to be due rather to their tangential than to their lateral ffness One remarkable instance of this effect of a boundary ssessing tangential stiffness is shown by the effect of a film of on the surface of water exposed to the wind The oil film erts a very small but appreciable tangential constraint, with the sult that the motion of the water below the film tends to become stable This results in the formation of eddies below the surface, d the energy, which is otherwise imparted by the action of the wind form and maintain stable wave motion, is now absorbed in the stitution of eddy motion, with the well-known effect as to the lling of the waves.

Where two streams of fluid are moving with different velocities common surface of separation is in a very unstable condition ynolds showed this by allowing the two liquids, carbon bisulide and water, to form a horizontal surface of separation in a g horizontal tube. The tube was then slightly tilted so as to produce elative axial motion of the fluids, when it was found that the tion was unstable for extremely small values of the relative ocity.

This also explains why diverging boundaries are such a cause turbulence. Experiment shows that in such a case as shown in

*Phil. Trans. Roy. Soc., 1883.

fig. 2 the high-velocity fluid leaving the pipe of small section is projected as a core into the surrounding mass of dead water, thereby giving rise to the conditions necessary for eddy formation.

More recent experiments* tend to show that the foregoing conclusions as to the effect of the curvature of the paths in affecting the manner of motion, are only true where the outer boundary of

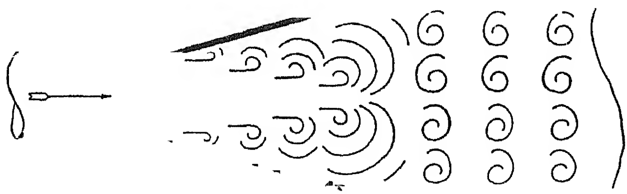

Fig 2

the fluid is formed by a solid surface, and that in some cases—as shown at the impact of a steady jet on a plane surface, at the efflux of a jet from a sharp-edged orifice, and in motion in a free vortex—curved motion, with the velocity greatest at the inside and not at the outside of the curve, tends to stream-line motion. Generally speaking, wherever the velocity of flow is increasing and the pressure diminishing, as where lines of flow are converging, there is an overwhelming tendency to stability of flow In a tube with converging boundaries it is this which leads to stability, and it is because this effect is sufficiently great to overcome the tendency to turbulent motion to which all solid boundaries, of whatever form, give rise, that the motion in such tubes is stable for very high velocities.

Hele Shaw's Experiments

The fact that stream-line motion is possible at fairly high velocities between parallel boundaries if the fluid is viscous, and if the distance between the boundaries is small, has been taken advantage of by Dr. Hele Shaw,† who produced stream-line motion in the flow of glycerine between two parallel glass plates, and showed the form of the stream lines by introducing coloured dye solution at a number of points. By inserting obstacles between the glass plates the form of the stream lines corresponding to flow

* *Memoirs, Manchester Lit and Phil Soc.*, 55, 1911, No. 13.

† *Trans. Inst. Naval Architects*, 1898, p. 27.

ough a passage or around a body of any required shape can ıs be obtained (figs. 3 and 4).

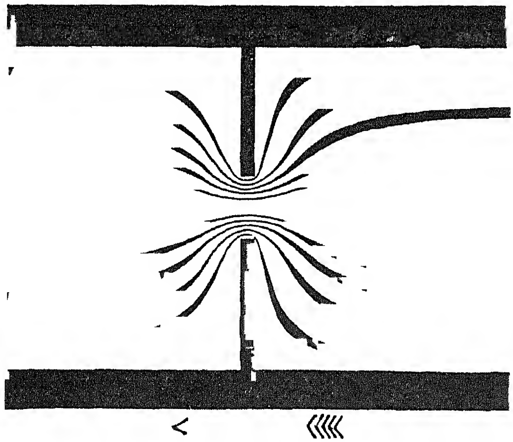

Fig 3

The form of the lines to be expected in the case of two-dimensional v of a perfect non-viscous fluid around bodies of simple and

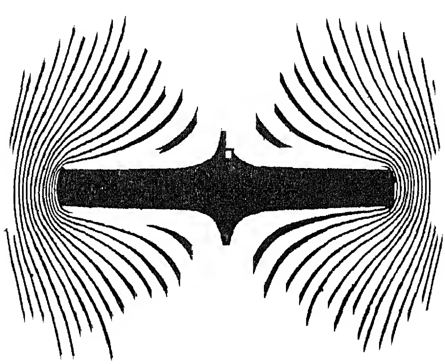

Fig 4

symmetrical shape, may be calculated,* and an examination of the stream lines obtained in the Hele Shaw apparatus shows that they are identical in form with those thus obtained by calculation, in spite of the fact that in one case the forces operating are entirely due to inertia, and in the other to viscosity. It has been shown by Sir George Stokes† that this is to be expected, for if PQ and P′Q′ (fig. 5) be two boundaries of a stream tube, and if PP′ and QQ′ be normals to one of the boundaries, ultimately these will become elements of two consecutive equipotential lines, and if produced will meet on the centre of curvature of the tube, so that if v and $v + \delta v$ be the velocities at P′ and P, and if r be the curvature and t the thickness,

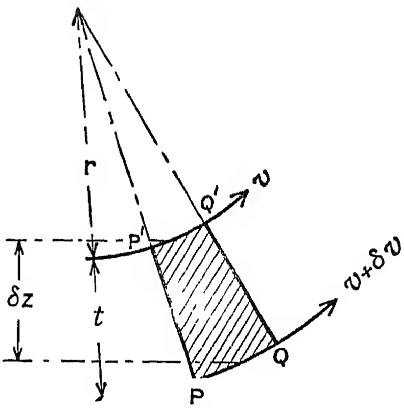

Fig 5

$$\frac{v + \delta v}{v} = \frac{\text{PQ}}{\text{P}'\text{Q}'} = \frac{r + t}{r};$$

$$\therefore \frac{\delta v}{t} = \frac{v}{r} \qquad \ldots \quad \ldots \qquad .(1)$$

Again considering the equilibrium of the element P′Q′QP, now imagined as part of a perfect non-viscous fluid, the centrifugal force will be balanced by the difference of normal pressures (δp) on the inner and outer faces, and by the resolved part of the difference of pressure due to the difference of level (δz) between the two faces If directions towards the centre of curvature be called positive, on resolving normally,

$$\frac{\rho v^2 t}{r} = -\delta p - \rho \delta z \ldots\ldots\ldots\ldots \ldots .(2)$$

On substituting for $\frac{vt}{r}$ from (1) this becomes

$$\rho v \delta v + \delta p + \rho \delta z = 0$$

or $$\frac{\rho v^2}{2} + p + \rho z = \text{constant},$$

* *Hydrodynamics*, Lamb, p 61, also *Trans. Inst N. A.*, 1898.
† *British Association Reports*, 1898, pp. 143–4.

hich is Bernoulli's equation of energy for a perfect non-viscous uid. It follows that the velocity relationship indicated in (1), p. 164, hich obtains when viscosity is the dominating factor, is also onsistent with the stream-line flow of a non-viscous fluid.

Critical Velocity

The nature of the two modes of fluid motion was first demonrated by Osborne Reynolds* in a series of experiments on parallel ass tubes of various diameters up to 2 in. These were fitted with ell-mouthed entrances and were immersed horizontally in a tank

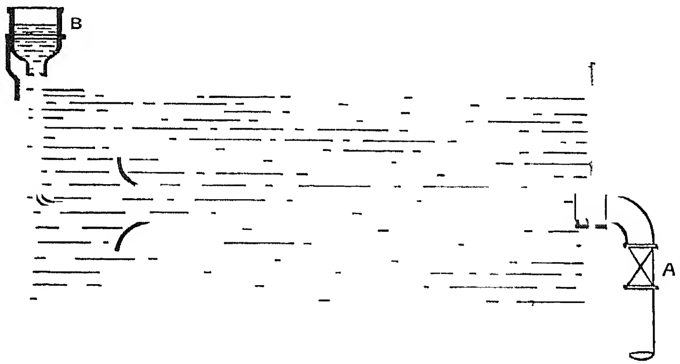

Fig 6

 water having glass sides (fig. 6) In these experiments the water the tank was allowed to stand until motionless. The outlet valve was then opened, allowing water to flow slowly through the tube. little water coloured with aniline dye was introduced at the trance to the tube through a fine tube supplied from the vessel B

At low velocities this fluid is drawn out into a single colour and extending through the length of the tube. This appears to e motionless unless a slight movement of oscillation is given to e water in the supply tank, when the colour band sways from side side, but without losing its definition. As the velocity of flow is adually increased, by opening the outlet valve, the colour band ecomes more attenuated, still, however, retaining its definition, ntil at a certain velocity eddies begin to be formed, at first interittently, near the outlet end of the tube (fig. 7). As the velocity still further increased the point of eddy initiation approaches the

* *Phil. Trans. Roy. Soc*, 1883.

mouthpiece, and finally the motion becomes sinuous throughout. The apparent lesser tendency to eddy formation near the inlet end of the tube is due to the stabilizing influence of the convergent mouthpiece.

The velocity at which eddy formation is first noted in a long

Fig 7

tube in such experiments is termed the " higher critical velocity ". There is also a " lower critical velocity ", at which the eddies in originally turbulent flow die out, and this is, strictly speaking, the true critical velocity. It has a much more definite value than the higher critical velocity, which is extremely sensitive to any disturbance, either of the fluid before entering the tube, or at the entrance. Over the range of velocities between the two critical values, the

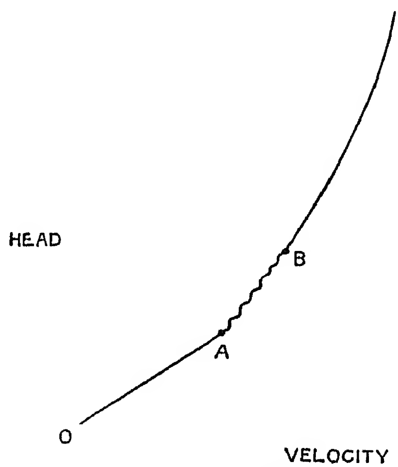

Fig. 8

fluid, if moving with stream-line flow, is in an essentially unstable state, and the slightest disturbance may cause it to break down into turbulent motion.

The determination of the lower critical velocity is not possible by the colour-band method, and Reynolds took advantage of the

'act that the law of resistance changes at the critical velocity, to letermine the values by measuring the loss of head accompanying lifferent velocities of flow in pipes of different diameters. On plotting a curve showing velocities and losses of head (fig. 8) it is found hat up to a certain velocity, A, for any pipe, the points lie on a ,traight line passing through the origin of co-ordinates. From A o B there is a range of velocities over which the plotted points are 'ery irregular, indicating general instability, while for greater velocities the points lie on a smooth curve, indicating that the loss of iead is possibly proportional to v^n.

To test this, and if so to determine the value of n, the logarithms

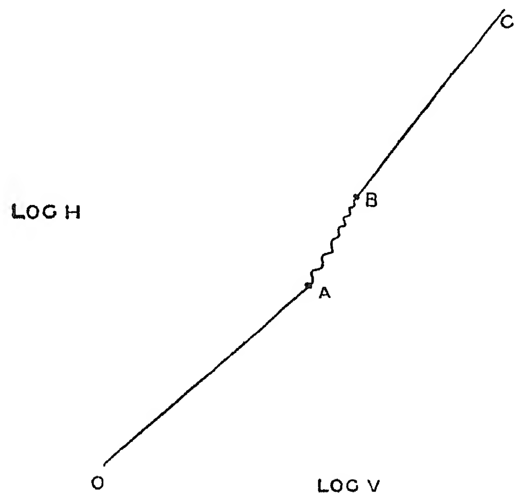

Fig 9

of the loss of head h and of the velocity were plotted (fig. 9) Then if

$$h = k v^n,$$
$$\log h = \log k + n \log v,$$

he equation to a straight line inclined at an angle $\tan^{-1} n$ to the ixis of log v, and cutting off an intercept log k on the axis of log h.

On doing this it is found that if the velocity is initially turbulent the plotted points lie on a straight line up to a certain point A, the value of n for this portion of the range being unity. At A, which marks the lower critical velocity, the law suddenly changes and h increases rapidly. There is, however, no definite relationship between h and v until the point B is reached. Above

this point the relationship again becomes definite, and within the limits of experimental error, over a moderate range of velocities the plotted points lie on a straight line whose inclination varies with the roughness of the pipe walls. The values of n determined in this way by Reynolds are:

Material of Pipe.	n
Lead . . .	1·79
Varnished	1·82
Glass	1·79
New Cast Iron . .	1·88
Old Cast Iron . .	2·0

those for cast iron being deduced from experiments by Darcy.

When tested over a wide range of velocities, it is found that the value of n in the case of a smooth-walled pipe is not constant but increases somewhat as the velocity is increased.

Between A and B the value of n is greater than between B and C, and the increased resistance accompanying a given change in velocity is greater even than when the motion is entirely turbulent. This is due to the fact that within this range of velocities eddies are being initiated in the tube, and the loss of head is due not only to the maintenance of a more or less uniform eddy regime, but also to the initiation of eddy motion.

Messrs Barnes and Coker* have determined the critical velocity in pipe flow by allowing water to flow through the given pipe which was jacketed with water at a higher temperature. The temperature of the water discharging from the pipe was measured by a delicate thermometer. So long as the motion is non-sinuous, transmission of heat through the water is entirely due to conduction and is extremely slow, so that the thermometer gives a steady reading sensibly the same as that in the supply tank. Immediately the critical velocity is attained, the rate of heat transmission is increased due to convection, and the change from stream-line to turbulent motion is marked by a sudden increase in the temperature of the discharge.

* *Proc. Roy. Soc.* A, 74.

The law governing the relationship between the critical velocity l the factors involved was deduced by Reynolds from a considera-1 of the equations of motion: for if the state of motion be supposed depend on the mean velocity in the tube and on the diameter, acceleration may be expressed as the difference of two terms, : of which is of the nature $\mu v/d$, and the other of the nature ρv^2. was then inferred that since the relative value of these terms bably determines the critical velocity, the latter will depend on ie particular value of the ratio $\mu/\rho v d$ To test the accuracy of ; conclusion experiments were made on pipes of different dia-ters, and with different values of μ obtained by varying the perature of the water between 5° C. and 22° C.

The results of the experiments fully justified the foregoing clusions, and showed that the critical velocity in a straight allel pipe is given by the formula

$$v_k = \frac{\mathrm{P}}{bd},$$

ie b is a numerical constant, and where $\mathrm{P} \propto \mu/\rho$. If the unit of gth is the foot, b equals 25·8 for the lower critical velocity, and ; for the higher velocity, while if t = temperature in degrees itigrade,

$$\mathrm{P} = \frac{1}{1 + 0\cdot 03368t + 0\cdot 000221t^2}$$

More recent experiments by Coker, Clement, and Barnes* and ers carried out by Ekman† on the original apparatus of Reynolds, w that by taking the greatest care to eliminate all disturbance ntry to the tube, values of the higher critical velocity considerably iter than (up to 3·66 times as great as) those given by the above nula may be obtained. The probability is, in fact, that there is definite *higher* critical velocity, but that this always increases 1 decreasing disturbances.

A general expression for the lower critical velocity in a parallel :, applicable to any fluid and any system of units, is

$$v_k = \frac{2300\mu}{d\rho} \quad \ldots \quad \ldots \quad \ldots (3)$$

$$= \frac{2300\nu}{d}.$$

* *Trans Roy Soc*, 1903, *Proc. Roy. Soc* A, 74.
† "Arkiv for Matematic" *Ast. Och. Fys*, 1910, 6, No 12.

Thus for water at 0° C., $\mu/\rho = \nu = 1{\cdot}92 \times 10^{-5}$ in foot-pou second units, so that

$$v_k = \frac{0{\cdot}0442}{d} \text{ ft.-sec., where } d \text{ is in feet.}$$

While for air at 0° C.,

$$\nu = 14{\cdot}15 \times 10^{-5};$$

$$\therefore\ v_k = \frac{0{\cdot}326}{d} \text{ ft.-sec. where } d \text{ is in feet.}$$

In this connection fig. 10* is of interest, as showing the resul of experiments on a number of pipes of different diameters, wi air and water flow, in which values of $R/\rho v^2$ are plotted as ordinat against the corresponding values of vd/ν or of $\log(vd/\nu)$ Here R the surface friction per unit area of the pipe wall. The curve consis of two parts connected by a narrow vertical band correspondi to a value of vd/ν of approximately 2300, over which the points fo the various pipes are somewhat irregularly disposed This ban indicates the range of instability between stream line and true tur bulent flow. The left-hand curve, corresponding to speeds belo the critical value, is calculated from the formula

$$R = \frac{8\mu v}{d},$$

theoretically corresponding to stream-line flow † It will be seen tha the points for both air and water flow lie closely on this curve, an that the break-down of the stream-line motion takes place in a cases at approximately the same value of vd/ν

As may be shown by an application of the principle of dynamica similarity,‡ formula (3) is a particular case of the general formul

$$v_k = \frac{k\nu}{l},$$

which is applicable to all cases of fluid motion. Here l is the lengt of some one definite dimension of the body The value of th constant k now depends only on the form of the surfaces over whicl flow is taking place. Thus in flow past similar plates immersed i water and in air, Eden § has shown by visual observation that th

* Stanton and Pannell, *Phil Trans Roy Soc* A, 214.
† Chap V, p 200 ‡ Chap V, p 193
§ *Advisory Committee for Aeronautics*, T.R., 1910–11, p 48.

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›e of flow, especially in the rear of the plate, is identical for identical ues of vl/ν, where l is the length of any particular side of the ıte.

Critical Velocity in Converging Tubes

In a converging tube the angle of convergence of the sides has arge effect on the critical velocity. At all ordinary velocities the tion in tubes or nozzles having more than a few degrees of congence may be considered as non-sinuous. Experiments on the v of water through circular pipes having sides converging unimly at an angle θ gave the following approximate values for the er critical velocity, at 14° C.*

	θ	5 Deg	7 5 Deg	10 Deg.	15 Deg
ıtical locity, -sec	At large section (3 in diameter)	1 5	1 94	2 44	3 25
	At throat ($1\frac{1}{2}$ in diameter) .	6 0	7 76	9 77	12 9
	At mean section ($2\frac{1}{4}$ in diameter)	2 7	3 45	4 34	5 73

› lower critical velocity in a $1\frac{1}{2}$-in. parallel pipe at this temperature 20 ft. per second. Should the ratio of higher to lower critical cities have the same value in a conical pipe as in a parallel pipe, would mean that in the case of a $1\frac{1}{2}$-in jet discharging from a verging nozzle with steady flow in the supply pipe, the critical city would have the following values

θ	5 Degrees.	7 5 Degrees.	10 Degrees	15 Degrees
tical velocity, ft -sec	39	50	63	84

n flow through a pipe bend, the velocity at which the resistance es to obey the laws of laminar flow is less than in a straight pipe. re is now a considerable range of velocity over which the resise is proportional to a power of the velocity higher than unity, n which turbulence is not developed. This is due to the develop-

* Gibson, *Proc. Roy. Soc.* A, 83, 1910, p. 376.

ment of a cross circulatory current superposed on the laminar flow. The critical velocity is not well defined but experiments * indicate that full turbulence is developed at a somewhat higher velocity than in straight pipes.

The Measurement of the Velocity of Flow in Fluids

Several methods are available for measuring the flow of fluids in pipes. Of these, the use of the Venturi meter or of the Pitot tube are the most common. Recent investigations into the possibilities of the hot-wire anemometer have shown that this is capable of giving excellent results, and that it is likely to be especially valuable for the measurement of pulsating flow.

The Venturi Meter

The Venturi meter, invented by Clemens Herschel in 1881, affords perhaps the simplest means of measuring the flow of a liquid. When fitted to a pipe line of diameter greater than about 2 in. its indications are, under normal conditions, thoroughly reliable so long as the

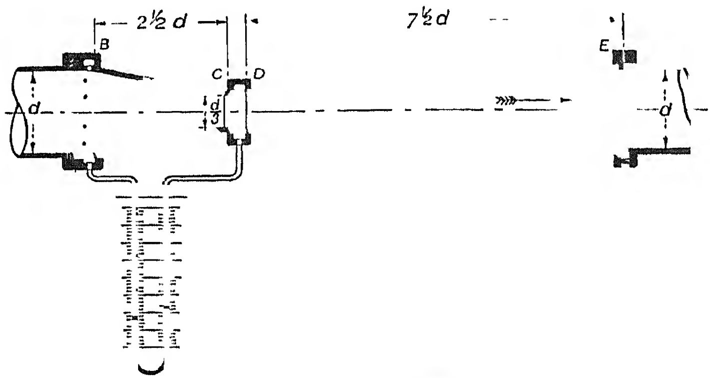


Fig. 11.—Venturi Meter

velocity in the pipe line exceeds 1 ft. per second, and the discharge may then be predicted, even without calibration, to within 1 or 2 per cent.

The meter is usually constructed of approximately the proportions shown in fig. 11, and consists essentially of an upstream cone usually having an angle of convergence of about 20°, connected to

* C. M. White, *Proc. Roy. Soc.* A, **123**, 1929, p. 645.

downstream cone whose angle of divergence is about 5° 30′, by asy curves. One annular chamber surrounds the entrance to the neter, and a second surrounds the throat, the mean pressures at hese points being transmitted to these chambers through a series f small holes in the wall of the pipe. The two chambers are conected to the two limbs of a differential pressure gauge which records heir difference of pressure h in feet of water. For this purpose a J-tube containing mercury may e used as in fig. 11. In this ase if the connecting pipes are ull of water it may readily be hown that the difference of ressure in feet of water is equal o 12·59 times the difference of evel of the tops of the mercury olumns By using an inverted J-tube with compressed air upplied to the highest portion f the tube, the difference of ressure may be directly reorded in feet of water When n automatic record is desired, he type of mechanism shown in g 12 may be used.

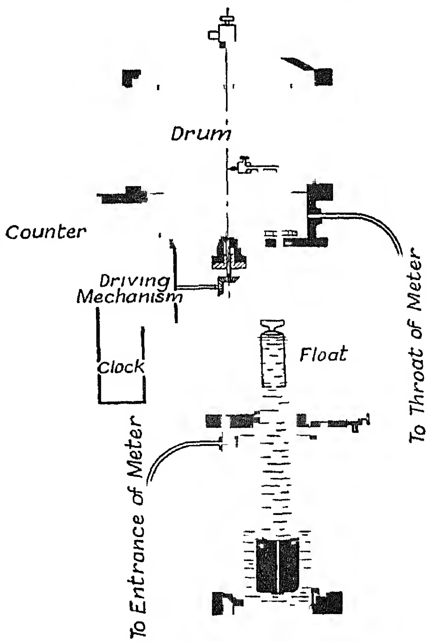


Fig 12 —Recording Mechanism for Venturi Meter

If P, A, V and p, a, v reresent the pressures in pounds er square feet, the areas in quare feet, and the mean velocities in feet per second respecvely at the entrance and throat of a meter whose axis is horiontal, neglecting any loss of energy between entrance and throat, ernoulli's equation of energy becomes

$$\frac{P}{W} + \frac{V^2}{2g} = \frac{p}{W} + \frac{V^2}{2g}\left(\frac{A}{a}\right)^2;$$

$$\therefore \frac{P-p}{W} = \frac{V^2}{2g}\left\{\left(\frac{A}{a}\right)^2 - 1\right\}$$

$$\text{or} \qquad V = \sqrt{\frac{2gh}{\left(\frac{A}{a}\right)^2 - 1}} \text{ ft.-sec.} \quad \ldots\ldots (4)$$

Actually owing mainly to frictional losses the velocity is slightly less than is indicated by formula (4), and is given by

$$V = C\sqrt{\frac{2gh}{\left(\frac{A}{a}\right)^2 - 1}} \text{ ft.-sec.}, \ldots\ldots\ldots\ldots(5)$$

where C varies from about 0·96 to 0·995,* usually increasing slightly with the size of meter. When used to measure pulsating flow, the value of C is reduced. The effect is, however, small for any such percentage fluctuations of velocity as are usual in practice, even with the discharge from a reciprocating pump. For accurate results the meter should be installed in a straight length of pipe removed from the influence of bends. Such bends set up whirling flow in the pipe, and this tends to increase the effective value of C.

The Venturi meter may also be used to measure the flow of gases.† In this case, for air, the discharge is given by

$$Q = CA\beta\sqrt{2gP_1W_1}, \ldots(6)$$

where P_1 is the pressure at entrance in pounds per square foot,
W_1 is the weight per cubic foot at P and temperature T;
and where, if p_1 is the pressure at entrance in pounds per square inch,
p_2 is the pressure at throat in pounds per square inch,
m is the ratio of areas at entrance and throat,
n is the index of expansion (1·408 for dry air expanding adiabatically),

$$\beta = \sqrt{\frac{\frac{n}{n-1}\left\{1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right\}\left(\frac{p_2}{p_1}\right)^{\frac{2}{n}}}{m^2 - \left(\frac{p_2}{p_1}\right)^{\frac{2}{n}}}}.$$

If τ_1 be the absolute temperature at entrance on the Fahrenheit scale, on writing $W_1 = \frac{2{\cdot}70p_1}{\tau_1}$ (the value for dry air) (6) reduces to

$$Q = 1{\cdot}10Ca_1\beta\frac{p_1}{\sqrt{\tau_1}} \text{ lb. per second}, \ldots\ldots\ldots(7)$$

where a_1 is the area at entrance in square inches.

* For a discussion of the variability of the coefficient C, see "Abnormal Coefficients of the Venturi Meter", *Proc Inst C E.*, **199**, 1914–5, Part I

† "Measurement of Air Flow by Venturi Meter", *Proc. Inst. Mech E.*, 1919, p 593, "Commercial Metering of Air, Gas, and Steam", *Proc. Inst. C. E.*, 1916–7, Part II, **204**, p. 108.

Experiments * indicate that the value of the coefficient C is not nstant, but that it diminishes as the ratio p_2/p_1 is increased approxi-ately as indicated in the following table

p_2/p_1	0·5	0 6	0·7	0·8	0·9	1·0
C	0 98	0·975	0·97	0·96	0·94	0·91

Measurement of Flow by Diaphragm in Pipe Line

The coefficients of discharge of standard sharp-edged orifices ;charging freely are known with a fairly high degree of accuracy, d where such an orifice can be used for measuring the steady flow her of a liquid or of air, the results may be relied upon as being

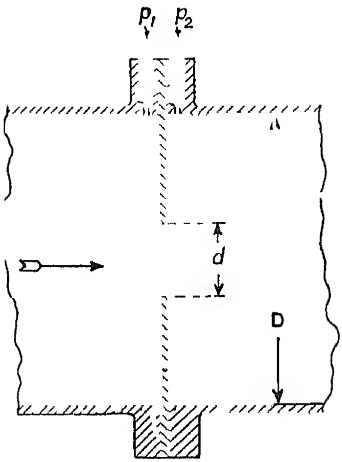


Fig 13

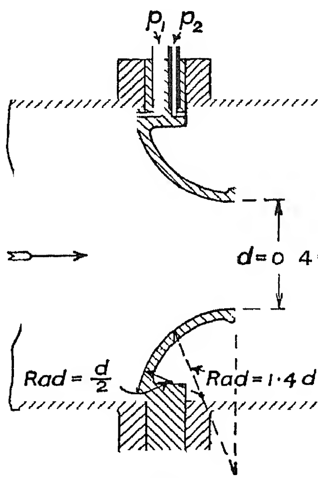


Fig 14

:urate within 1 or 2 per cent, if suitable precautions are taken. ving to the convenience of the method and the simplicity of the paratus, much attention has recently been paid to the use of fices through diaphragms in a pipe line for measuring the flow.

If D be the diameter of the pipe and d that of the orifice (fig. 13), ›dgson† states that the coefficient of discharge C for sharp-edged

* *Proc. Inst Mech E*, Oct 1919, p. 593.
† *Proc. Inst. C.E.*, 1916–17, Part II, p. 108.

orifice in a plate of thickness 0·02d, and for ratios of d/D less than 0·7 is 0·608 for water or air when p_2/p_1 is greater than 0·98, and is equal to 0·914 — 0·306 p_2/p_1 for air, steam, or gas when p_2/p_1 is less than 0·98, pressures being measured at the wall of the pipe immediately on each side of the diaphragm.

A rounded nozzle, if well designed, has a coefficient which varies from about 0·94 in small nozzles to 0·99 for large nozzles, either for water or air, if p_2/p_1 is greater than 0·6. Fig. 14* shows a form of nozzle in which the coefficient lies between 0·99 and 0·997.

The Pitot Tube

For measurements of the flow in pipes or in unconfined streams where the velocity is fairly high, the Pitot tube is capable of giving excellent results. This usually consists of a bent tube terminating in a small orifice pointing upstream, which is surrounded by a second tube whose direction is parallel to that of flow. A series of small holes in the wall of the outer tube admit water, at the mean pressure

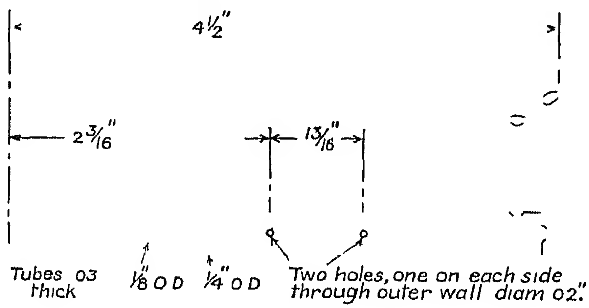


Fig 15

in their vicinity, to its interior, which is connected to one leg of a manometer. The other leg is connected to the central tube carrying the impact orifice. If v is the velocity of flow immediately upstream from this orifice, the pressure inside the orifice, where the velocity is zero, is equal to the sum of the statical pressure at the point, plus $kv^2/2g$ ft of water, where k is a constant whose value approximates closely to unity in a well-designed tube. It follows that the difference of level of the two legs of the manometer equals $kv^2/2g$.

Figs. 15, 16, and 17 show modern types of this instrument. Fig. 15 shows the type used for measuring the air speed of aeroplanes

* *Engineering*, 1st Dec , 1922, p. 690.

and for wind tunnel investigations. A tube of this type having the dimensions shown gave K = 1·00 within 1 per cent.*

The tube illustrated in fig. 16 gave a value of C = 0·926 when calibrated by towing through still water, and 0·895 when calibrated in a 2-in. pipe. The low value of C in still water is probably due to the fact that the pressure orifices are too near the shoulder of the pressure pipe. If this were lengthened, with the orifices farther back, the coefficient would probably be higher. The difference between the calibration in still water and in the small pipe is o be expected, since he velocity at the section of the pipe containng the pressure orifices s of necessity increased y the presence of the ube, and the pressure s recorded by the static ressure column will onsequently be less than n the plane of the impact orifice. This effect will increase with the atio of the diameter of ube to that of the pipe, nd unless this ratio is

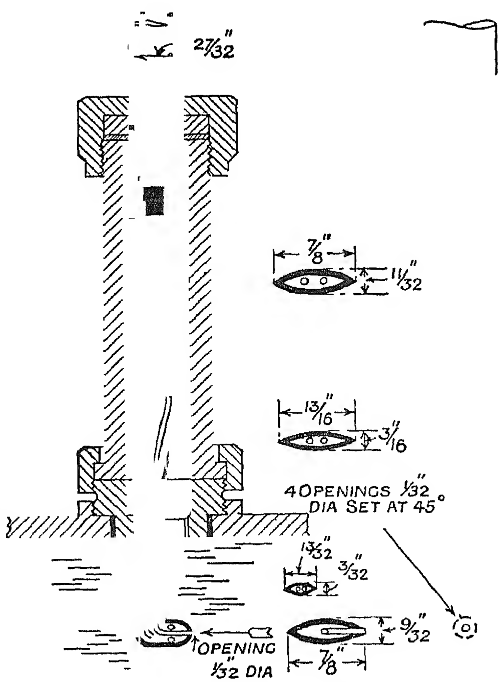


Fig 16 —Details of Pitot Tube

mall, the tube should be calibrated in a pipe of approximately the ame dimensions as that in which it is to be used. It is preferble, for pipe work, to use a simple Pitot tube having only an npact orifice, and to obtain the static pressure from an orifice in e pipe walls in the plane of the impact orifice. Using the tube this way, the coefficient C may usually be taken as 0 99, within per cent.

In the type of Pitot tube shown in fig 17, and known as the Pitometer", the pressure at the downstream orifice is less than the

* "The Theory and Development of the Pitot Tube", Gibson, *The Engineer*, 7th July, 1914, p. 59.

statical pressure in the pipe and the coefficient is less than in the normal type. For the tube shown, calibrations in flowing water in pipes give a mean value of C = 0·916. Owing to eddy formation at the downstream orifice the coefficient of such a tube fluctuates within fairly wide limits.

The Pitot tube may be calibrated either in still water or in a current. In the latter case the mean velocity is computed from readings taken at a large number of points in a cross section, and

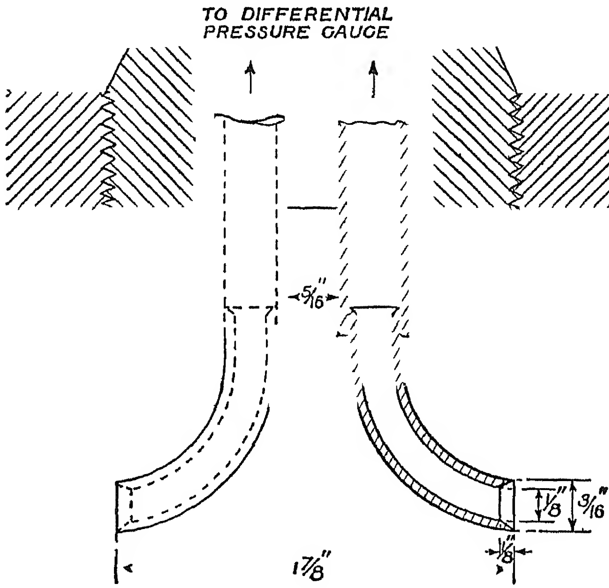


Fig 17

the coefficient of the instrument is adjusted so as to make this mean velocity agree with that obtained from weir flow, current meter, or gravimeter measurements on the same stream.

Without exception, observers have found that a still-water rating gives a somewhat higher value of C. The explanation would appear to be twofold. In the first place, the velocity of flow in a moving current is never quite steady, but suffers a series of periodic fluctuations, and since the Pitot tube is an instrument which essentially measures the mean momentum, or the mean (v^2) of the flow and not its mean velocity, any such fluctuation superposed on a given mean velocity will give an increased head reading. In the second place, when metering a flowing current the average tube cannot be used at

ɔoints very near the boundary where the velocity is least, and for this eason also the mean recorded velocity tends to be too high.

It follows that although a still-water or still-air rating represents he true coefficient of the instrument, this requires to be reduced omewhat for use in a current, the effect increasing with the unteadiness of the current. Where a high degree of accuracy is equired, the rating should be carried out under conditions as nearly s possible resembling those under which measurements have afterwards to be made.

For measurements of the flow in pipes, the instrument should e used if possible at a section remote from any bend or source f disturbance. For approximate work the velocity of the central lament may be measured. This when multiplied by a coefficient hich varies from 0·79 in small pipes to 0·86 in large pipes gives ie mean velocity Alternatively the velocity may be measured at ie radius of mean velocity, which varies from 0·7a in small pipes 0·75a in large pipes, where a is the radius of the pipe. These ilues, however, only apply to a straight stretch of the pipe, and it is necessary to make measurements near a bend, and in any case r accurate results, the pipe should be traversed along two diaeters at right angles, and the velocities measured at a series of dii If δr is the width of an elementary annulus containing one ries of such measurements whose mean value is v, the discharge then given by

$$Q = \int_0^a 2\pi r v dr$$

Several methods are available for determining the mean velocity flow of a liquid in an open channel. This may be obtained:

(*a*) by the use of current meters giving the velocity at a series points over a cross section of the channel;

(*b*) by the use of a standard weir;

(*c*) by the use of floats;

(*d*) by chemical methods. This method is best adapted to rapid d irregular streams, although it may be applied to the measureent of pipe flow. It consists in adding a solution of known strength some chemical for which sensitive reagents are available, at a iform and measured rate into a stream,* and by collecting and

* For a description of the method of introducing the solution uniformly, refere may be made to *Mechanical Engineering*, **44**, April, 1922, p. 253, or to *dro-electric Engineering*, Gibson, Vol. I, p. 29.

analysing a sample taken from the stream at some lower point where admixture is complete. The solution should be added and the sample taken at a number of points distributed over the cross section. Various chemicals may be used Unless the water is distinctly brackish, common salt is suitable. If brackish, sulphuric acid or caustic soda may be used. With a solution consisting of 16 lb. of salt per cubic foot of water, a dilution of 1 in 700,000 will give, on titration with silver nitrate, a precipitate weighing 1 mgm. per litre of the sample, and the gravimetric analysis of such a sample will enable an accuracy of 1 per cent to be attained.

If Q = discharge of stream to be measured in cubic feet per second,
q = quantity of solution introduced in cubic feet per second,
C_1 = concentration of salt in the natural stream water in pounds per cubic foot,
C_2 = concentration of salt in the sample taken downstream in pounds per cubic foot,
C = concentration of salt in the dosing solution in pounds per cubic foot,

then $$Q = \left(\frac{C - C_2}{C_2 - C_1}\right)q,$$

and if V_1, V_2, and V are the volumes of silver nitrate solution respectively necessary to titrate unit volume of normal stream water, of the downstream sample, and of the dosing solution,

$$Q = \left(\frac{V - V_2}{V_2 - V_1}\right)q.$$

(*e*) By injecting colour and by noting the time required for this to cover a measured distance.

A close approximation to the true discharge may be obtained either by the use of a weir, of current meters, or by chemical methods if suitable precautions are taken.* The colour method would not, in general, appear to be so reliable, and float measurements cannot be relied upon for any close degree of accuracy.

The Effect of Fluid Motion on Heat Transmission

Apart from the effect of radiation, the heat transmission between a solid surface and a fluid in motion over it will, for a given difference in temperature, be proportional to the rate at which the fluid

* See *Hydraulics*, Gibson, 1912 (Constable & Co), p 346.

articles are carried to and from the surface, and therefore to the iffusion of the fluid in the vicinity of the surface. Such diffusion epends on the natural internal diffusion of the fluid at rest, and n the eddies produced in turbulent motion which continually bring ·esh particles of fluid up to the surface In stream-line flow the econd source of diffusion is absent; the heat transmission can only ike place in virtue of the thermal conductivity of the fluid; and ie rate of heat transmission is very small. Assuming that H, the eat transmitted per unit time per unit of surface, is proportional) θ, the difference of temperature between surface and fluid, the ombined effect of the two causes may be written

$$H = A\theta + B\rho v\theta, \quad \ldots\ldots \quad (8)$$

here ρ is the density of the fluid, A and B are constants depending n its nature, and v is its mean velocity

As pointed out by Reynolds,* the resistance to the flow of a uid through a tube may be expressed as

$$R = A'v + B'\rho v^2, \quad \ldots\ldots \quad (9)$$

id various considerations lead to the supposition that A and B ı (8) are proportional to A′ and B′ in (9) For assuming, as is now enerally accepted, that even in turbulent flow there is a thin layer f fluid at the surface which is in stream-line motion, the heat trans-ıission through this layer will be by conduction, and from the oundary of this layer to the main body of fluid by eddy convec-on In stream-line flow the transfer of momentum which gives se to the phenomena of viscosity is due to internal diffusion, while ı turbulent motion the transference of momentum is due to eddy onvection, so that it would appear that the mechanism giving rise) resistance to flow is essentially the same as that giving rise to eat transmission, both in stream-line and turbulent motion

The following general explanation of the Reynolds law of heat ansmission is due to Stanton † Neglecting the effect of conduc-vity compared with that of viscosity, the ratio of the momentum ›st by skin friction between two sections δx in apart, to the total ıomentum of the fluid, will be the same as the ratio of the heat ctually supplied by the surface, to that which would have been ıpplied if the whole of the fluid had been carried up to the surface.

* Reynolds, *Manchester Lit and Phil Soc.*, 1834

† "Note on the Relation between Skin Friction and Surface Cooling", *Tech. eport, Advisory Committee for Aeronautics*, 1912–3.

Thus in pipe flow:

if δp is the difference in pressure at the two sections;
δT is the rise in temperature between the two sections;
W is the weight of fluid passing per second;
v_m is the mean velocity of flow;
T_m is the mean temperature of the fluid;
T_s is the temperature of the surface;
a is the radius of the pipe;

the above relationship becomes

$$\frac{\delta p(\pi a^2)}{\frac{W}{g}v_m} = \frac{W\delta T}{W(T_m - T_s)} \quad \ldots\ldots\ldots\ldots\ldots(10)$$

The heat gained per unit area of the pipe per second is

$$\frac{\sigma\frac{W}{g}\delta T}{2\pi a\delta x},$$

where σ is the specific heat. If R is the resistance per unit area,

$$R = \frac{\pi a^2}{2\pi a}\frac{\partial p}{\partial x},$$

so that if H is the heat transmitted per unit area per second,

$$H = \frac{R\sigma(T_s - T_m)}{v_m} \quad \ldots \quad \ldots \quad \ldots(11)$$

Since, as pointed out by Reynolds, the heat ultimately passes from the walls of the pipe to the fluid by conductivity, a correct expression for heat transmission should involve some function of the conductivity, and for this reason expression (11) can only be expected to give approximate results. In spite of this it enables some results of extreme practical value to be deduced. Thus if $R = kv^n$, and if σ be assumed constant,

$$H \propto v^{n-1}(T_s - T_m), \quad \ldots\ldots\ldots\ldots\ldots(12)$$

so that if the resistance be proportional to v^2, and if $T_s - T_m$ be maintained constant, the heat transmitted per unit area will be proportional to v, and since the mass flow is also proportional to v, the change in temperature of the fluid during its passage through

the pipe will be independent of the velocity. Otherwise, the heat transmitted will be directly proportional to the mass flow.

The general truth of this was demonstrated experimentally by Reynolds,* who showed that when air was forced through a hot tube, the temperature of the issuing air was sensibly independent of the speed of flow.

In the case of the flow of hot gases through the tubes of a boiler, or of the water through the tubes of a condenser, n is usually less than 2 and has a value of about 1·85. Moreover, in the former case any increase in the velocity of flow will be accompanied by an increase in the temperature of the metal surface, so that for both reasons the heat transmitted is not quite proportional to the mass flow, and the issuing gases are slightly hotter with a high velocity than with a low velocity of flow. The difference is, however, not great, and it appears that by increasing the velocity of flow of the fluid, the output of a steam boiler, or of a surface condenser, may be considerably increased without seriously affecting the efficiency. This is in general accordance with Nicholson's † investigations on boilers working under forced draught. These showed that by increasing the mass flow, the heat transmitted to the water was increased in almost the same proportion, while the temperature of the flue gases was only slightly increased.

Numerous other observers ‡ have verified the general truth of the relationship expressed in equation (12), p 182, for the flow of liquids and gases through pipes. Its more general application to other cases of heat dissipation in a current still awaits experimental proof.

Experiments on the heat dissipation from hot wires of small diameter in an air current, show that this is proportional to $v^{0.5}$, which, if this relationship is correct, would indicate that the resistance should be proportional to $v^{1.5}$. This is not in agreement with the generally accepted result that the resistance is proportional to v^2. An examination of the experimental data shows, however, that the product of vd in the wires on which the heat measurements were made, was small, and an examination of the curve showing $R/\rho v^2 d^2$ plotted against vd/ν (fig. 4, Chap. V), shows that in this part of the

* *Memoirs, Manchester Lit and Phil. Society*, 1872

† "Boiler Economics and the Use of High Gas Speeds", *Trans. Inst of Engineers and Shipbuilders in Scotland*, 54; "The Laws of Heat Transmission in Steam Boilers", J T Nicholson, D.Sc, *Junior Institute of Engineers*, 1909.

‡ Stanton, *Phil. Trans. Roy Soc.* A, 190, 1897, Jordan, *Proc. Inst. Mech. Engineers*, 1909, p 1317, Nusselt, *Zeitschrift des Vereines deutscher Ingenieure*, 23rd and 30th Oct, 1909.

range the curve is very steep, indicating that the resistance is proportional to a value of n much less than 2. Although the data are insufficient to indicate the exact value of n they do not, at all events, disprove the foregoing hypothesis

The difficulty in forming any definite decision as to the general validity of the hypothesis arises from the fact that in most resistance experiments on cooling systems, it has been tacitly assumed that the resistance is proportional to v^2, and the published data usually give the average value of the coefficient of resistance based on this assumption. Thus the resistance of honeycomb radiators is known to be nearly proportional to v^2, while the heat transmission per degree difference of temperature is approximately proportional to $v^{0\cdot 85}$.

Experiments by Stanton and by Pannell* show that while equation (11), p. 182, gives moderate results for air flow through pipes, the calculated results obtained with water as the fluid are very different from those deduced experimentally, as appears from Table I.

TABLE I

Pipe Dia, Cm	Mean Vel, Cm per Sec	Mean Temperature: Surface, Deg C	Mean Temperature: Fluid, Deg C	Friction, Dynes per Sq Cm	Heat Transmitted, Calories per Sq Cm per Sec	Value of $\frac{R\sigma(T_s-T_m)}{V_m}$	Fluid
0 736	296	28 2	15 93	29 8	4 43	12 35	Water
0 736	296	51 65	39 65	26 0	5·08	10 5	
1 39	123 2	47·2	20 96	50 6	5 36	10·8	
1·39	69 0	47 3	21 21	17 1	3·28	6·5	
4 88	940	36 2	22·7	3 18	0 0162	0 0109	Air.
4 88	1180	37 4	22 5	5 14	0 0205	0 0155	
4 88	1480	43 5	23 5	8 15	0 0300	0·0266	
4 88	2188	43 0	26·2	14 9	0 0369	0·0267	

It will be seen that in the case of air flow the heat transmission calculated from equation (11) is about 76 per cent of that observed, while for water the calculated value is twice as great as that observed.

* *Phil Trans. Roy Soc* A, 190, *Tech Report, Advisory Committee for Aeronautics*, 1912–3

Mr. G. I. Taylor * suggests that equation (11), p. 182, may be modified to take into account the effect of conductivity by assuming that there is a surface layer of thickness t, having laminar motion, through which heat is conveyed by conduction; that the velocity at the inner boundary of this layer is U, and the temperature T_1; and that between the centre and this layer heat transmission is due to eddy convection.

The temperature drop in the laminar layer, of conductivity k, is given by

$$T_s - T_1 = \frac{Ht}{k}.$$

If R is the resistance per unit area at the surface, this will be equal to the resistance to shear of the lamina.

$$\therefore\ R = \frac{\mu U}{t},$$

so that $$T_s - T_1 = \frac{H\mu U}{kR} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

By analogy with (11) the rate at which heat is transmitted from the layer by eddies is

$$H = \frac{R\sigma(T_1 - T_m)}{(v_m - U)} \ldots \quad \ldots (14)$$

Substituting for $\frac{H}{R}$ in (14) from (13) gives

$$\frac{T_1 - T_m}{T_s - T_1} = \frac{(v_m - U)}{U}\frac{k}{\sigma\mu}.$$

If $\frac{U}{v_m} = r$,

$$\frac{v_m - U}{U} = \frac{1 - r}{r};\quad \frac{v_m - U}{v_m} = 1 - r;\quad \frac{T_s - T_m}{T_1 - T_m} = 1 + \frac{r}{1 - r}\left(\frac{\mu\sigma}{k}\right).$$

$$\therefore\ H = R\sigma\frac{(T_1 - T_m)}{(v_m - U)}$$

$$= R\sigma\left(\frac{T_s - T_m}{v_m}\right)\left\{\frac{1}{1 + r\left(\frac{\mu\sigma}{k} - 1\right)}\right\} \ldots \quad (15)$$

This equation is identical with (11) if the quantity in brackets is

* Advisory Committee for Aeronautics, *Reports and Memoranda*, No. 272, 1916.

unity, i.e if $\mu\sigma = k$. For air this is very nearly the case, since $k = 1{\cdot}6\mu C_v$ where C_v is the specific heat at constant volume, and $C_v = \frac{\sigma}{1{\cdot}4}$, so that $\frac{\mu\sigma}{k} = 0{\cdot}88$. In the case of water, however, at 20° C., $\mu = 0{\cdot}01$, $k = 0{\cdot}0014$, $\sigma = 1{\cdot}0$, and $\frac{\mu\sigma}{k} = 7{\cdot}1$.

Stanton * has shown that the value of r necessary to bring the results as found from equation (15), p. 185, into line with the experimental results for water quoted in Table I, p. 184, is 0·29, and that similar experiments by Soennekei † require a mean value of 0·34. Taylor, from an examination of data by Lorentz,‡ concludes that the ratio is approximately 0·38. Some idea of its value in the case of air may be deduced from direct measurements by Stanton, Marshall, and Bryant,§ of the velocities in the immediate vicinity of the pipe wall. These measurements would appear to indicate that true laminar flow is instituted at a point where $\frac{U}{v_m}$ is approximately 0·14. They show, however, that it is erroneous to assume that at this point the change to true turbulent flow is abrupt, but that the change is gradual over an appreciable radial depth of the fluid. It follows that equation (15) has not a strictly rational basis, but that by assuming $r = 0{\cdot}30$ it gives results which are, for practical purposes, not seriously in error.

The ratio

$$\frac{\text{drop in temp in surface layer}}{\text{drop in temp. in rest of tube}} = \frac{T_s - T_1}{T_1 - T_m} = \left(\frac{r}{1 - r}\right)\frac{\mu\sigma}{k}.$$

Thus, if the effective value of $r = 0{\cdot}30$, the ratio is 3·0 for water at 20° C., and 0·38 for air.

Reynolds || has shown by an application of the principle of dynamical similarity that in the case of pipe flow

$$\frac{\partial p}{\partial x} = \frac{\nu^{2-n}}{d^{3-n}} v_m^n \frac{B^n}{A},$$

* *Dictionary of Applied Physics,* Vol I, p 401.

† *Konig Tech. Hochschule,* Munich, 1910

‡ *Abhandlung uber theoretische Physic,* Band, I, p. 343.

§ In air flow $R = 0{\cdot}002\rho V_m^2$ approximately, and Stanton's experiments (*Proc. Roy Soc.* A, 1920) indicate that t is approximately 0·005 cm. when $\mu = 0{\cdot}00018$ and $V_m = 1850$ cm. per second This makes $U = 0{\cdot}14V_m$.

|| Chapter V.

nd if this value of $\frac{\partial p}{\partial x}$ be used in (10), p. 182, on writing $N = \pi r^2 \rho v$, the expression becomes

$$\frac{dT}{dx} = \frac{B^n}{A}\frac{g}{\rho}\frac{\nu^{2-n}}{d^{3-n}} v_m^{n-2}(T_s - T), \ldots\ldots\ldots(16)$$

here now T is the temperature of the fluid, and $\frac{dT}{dx}$ is the temperature gradient along the pipe.

If T_s is sensibly constant along the pipe, integration of (16) ives

$$\log_e \frac{T_s - T_1}{T_s - T_2} = \frac{B^n}{A}\frac{g}{\rho}\frac{\nu^{2-n}}{d^{3-n}} v_m^{n-2} l, \ldots\ldots\ldots(17)$$

here l is the length of the pipe, and T_1 and T_2 are the temperatures the fluid at inlet and outlet.

Stanton,* from experiments on heat transmission from water a cold tube and vice versa, deduced the expression, for small llues of $T_1 - T_2$,

$$\log_e \frac{T_s - T_1}{T_s - T_2} = k\frac{\nu^{2-n}}{d^{3-n}} v_m^{n-2} l \{(1 + \alpha T_s)(1 + \beta T_m)\}, \quad .(18)$$

here, in C G S units, $\alpha = 0.004$ and $\beta = 0.01$ It will be noted at this expression is identical in form with (17), except for the st two factors, which were introduced to take into account the fect of the variation in conductivity, with temperature, of the surface m of water. In those experiments in which the heat flow was from etal to water, k had a mean value of 0·0104 With flow in the other rection k was, however, distinctly less, having a mean value of proximately 0·0075.

pplication of the Principle of Dimensional Homogeneity to Problems involving Heat Transmission

The principle of dimensional homogeneity, Chap. V,† may readily extended to problems involving heat transmission. In this case, addition to the three fundamental mechanical units, a thermal it is needed to define all the quantities involved. Taking temperae T as this unit, the new quantities, heat flow H, conductivity k,

* *Trans. Roy Soc.* A, 1897

† See also a note by Lord Rayleigh, *Nature*, 95, 1915, p. 66.

and specific heat σ which are now involved, may be expressed dimensionally as

$$H = \begin{Bmatrix}\text{heat flow per}\\ \text{unit time}\end{Bmatrix} = \begin{Bmatrix}\text{energy per}\\ \text{unit time}\end{Bmatrix} = ML^2t^{-3}.$$

$$k = \begin{Bmatrix}H \times \text{length}\\ \text{sectional area} \times T\end{Bmatrix} = \frac{H}{LT} = MLt^{-3}T^{-1}.$$

$$\sigma = \begin{Bmatrix}\text{heat per unit mass}\\ \text{rise in temperature}\end{Bmatrix} = \frac{Ht}{MT} = L^2t^{-2}T^{-1}.$$

If attention be confined to the large class of problems of practical importance, involving the transmission of heat between a fluid and a surface moving with relative velocity, where temperature differences are so small—not exceeding a few hundred degrees—that radiation is only of secondary importance, the only quantities involved are

$$H,\ T,\ k,\ \sigma,\ v,\ l,\ \rho,\ \mu$$

We select l, v, ρ and σ as the four independent quantities, and combine them with the other four H, T, k and μ, in turn, so as to obtain $8 - 4$ $(=4)$ dimensionless quantities K, as explained at p 198; i e. we write $H = l^x v^y \rho^z \sigma^n$, and similarly with T, k and μ We thus find

$$K_1 = \frac{H}{l^2v^3\rho};\ K_2 = \frac{\sigma T}{v^2};\ K_3 = \frac{k}{lv\rho\sigma};\ K_4 = \frac{\mu}{lv\rho},$$

whence $$\psi\left(\frac{H}{l^2v^3\rho},\ \frac{\sigma T}{v^2},\ \frac{k}{lv\rho\sigma},\ \frac{\mu}{lv\rho}\right) = 0;$$

or $$H = \rho l^2v^3\phi\left(\frac{\sigma T}{v^2},\ \frac{k}{lv\rho\sigma},\ \frac{\mu}{lv\rho}\right),$$

which, by combining the last two terms, becomes

$$H = \rho l^2v^3\phi\left(\frac{\sigma T}{v^2},\ \frac{k}{\sigma\mu},\ \frac{\mu}{lv\rho}\right) \quad \ldots\ldots\ldots\ldots (19)$$

At such speeds as are usual in the case of air flow over air-cooled engine cylinders, of the flow of gases through boiler flues, or of heating or cooling liquids through pipes, experiment shows that the heat flow is sensibly proportional to v^n, where n is between 0·5 and 1·0, its value depending on the type of flow and the form of surface. Experiment, moreover, indicates that if radiation be neglected the heat flow is directly proportional to the difference of temperature

etween the fluid and the surface, in which case the function ϕ in (19) ust be of the form $\frac{\sigma T}{v^2} F\left(\frac{k}{\sigma\mu}, \frac{\mu}{lv\rho}\right)$, and (19) becomes

$$H = \rho l^2 v\sigma T F\left(\frac{k}{\sigma\mu}, \frac{\mu}{lv\rho}\right). \quad \ldots\ldots\ldots\ldots (20)$$

, in addition, $H \propto v^n$, F must be of the form $\left(\frac{\mu}{lv\rho}\right)^{1-n} f\left(\frac{k}{\sigma\mu}\right)$, and 0) becomes

$$H = \rho^n l^{1+n} v^n \mu^{1-n} \sigma T f\left(\frac{k}{\sigma\mu}\right). \quad \ldots\ldots\ldots\ldots (21)$$

the fluid to which k, σ and μ belong is a gas, $f\left(\frac{k}{\sigma\mu}\right)$ is a constant, the kinetic theory of gases. In this case, we may take for $f\left(\frac{k}{\sigma\mu}\right)$ ther $A\frac{k}{\sigma\mu}$ or $B\left(\frac{k}{\sigma\mu}\right)^{1-n}$, these being constants; thus obtaining the ternative forms for H,

$$H \propto l^{1+n}\left(\frac{v\rho}{\mu}\right)^n kT,$$

$$\text{or} \qquad H \propto l^{1+n}(v\rho\sigma)^n k^{1-n} T \quad \ldots\ldots\ldots\ldots (22)$$

F is the total resistance to the steady motion of the fluid, then ice the heat loss per degree difference of temperature, per unit ecific heat, is of dimensions

$$\frac{H}{\sigma T} = \frac{ML^2t^{-3}}{L^2t^{-2}T^{-1}\ T} = Mt^{-1},$$

hile F is of dimensions MLt^{-2}, the ratio $F\Big/\frac{H}{\sigma T}$ is of dimensions $l/t = v$, so that the index n in $H \propto v^n$ should be less by unity than e corresponding index in $F \propto v^{n'}$ From this it appears that with w so turbulent as to give the n^2 law of resistance, n in equation 2) becomes 1, and

$$H \propto l^2 v\rho\sigma T, \quad \ldots\ldots\ldots\ldots (23)$$

ile with stream-line flow ($n' = 1$ and $n = 0$),

$$H \propto lkT. \quad \ldots\ldots\ldots\ldots (24)$$

r example, in the case of flow through similar pipes, where the erm may be taken to represent the diameter, equation (23) indites that in such circumstances H is independent of the conducity of the fluid. Also since the weight of fluid W passing

a given section per second is proportional to the product $d^2v\rho$

$$H \propto W\sigma T$$

It follows that in similar pipes the heat transmission per unit degree difference of temperature between wall and fluid is proportional to the weight of fluid passing, or in other words, that with a given inlet temperature the outlet temperature is independent of the weight of fluid

If, however, n is somewhat less than 1, as is usually the case in practice, equation (22) shows that

$$H \propto W \frac{\sigma^n k^{1-n}}{(dv\rho)^{1-n}} T$$

$$\propto W^n \sigma^n (kd)^{1-n} T,$$

so that with a given pipe and fluid, the heat transmission does not increase quite so fast as the mass flow, and the outlet temperature will increase somewhat as the flow is increased.

CHAPTER V

Hydrodynamical Resistance

A body in steady motion through any real fluid, or at rest in moving current, experiences a resistance whose magnitude depends ɔon the relative velocity, the physical properties of the fluid, the ze and form of the body, and, at velocities above the critical, also ɔon its surface roughness.

At velocities below the critical, where the flow is " stream line ", e resistance is due essentially to the viscous shear of adjacent layers the fluid. It is directly proportional to the velocity, to the vis- sity, and, in bodies of similar form, to the length of corresponding mensions Thus the resistances to the motion of small spheres such velocities are proportional to their diameters *

With stream-line motion there is no slip at the boundary of solid d fluid, and the physical characteristics of the surface do not ect the resistance.

At velocities above the critical, where the motion as a whole is finitely turbulent, there would still appear to be a layer of fluid contact with the surface in which the motion is non-turbulent † ıe thickness of this layer is, however, very small, and any increase the roughness of the surface, by increasing the eddy formation, creases the resistance. At such velocities the resistance is due in rt to the viscous shear in this surface layer, but mainly to eddy mation in the main body of fluid. This latter component of the sistance depends solely on the rate at which kinetic energy is being en to the eddy system, and is proportional to the density of the ıd and to the square of the velocity.

Although the viscosity of a fluid provides the mechanism by which dy formation becomes possible, and by which the energy of the dies, when formed, is dissipated in the form of heat, it has only very small effect on the magnitude of the resistance in turbulent otion, and, as will be shown later, it can have no direct effect in

* See H. S Allen, *Phil. Mag* , September and November, 1900.

† Stanton, *Proc Roy. Soc.* A, 97, 1920.

a system in which the resistance is wholly due to eddy formation and in which the resistance is, in consequence, proportional to v^2.

Experiments carried out over a limited range of velocities have usually shown that with turbulent flow the resistance of any given body is proportional to v^n, where n is slightly less than 2, although experiments on flow in rough pipes, on the resistance of cylinders, of inclined plane surfaces, and of air-ship bodies, show that in such cases the variation from the index 2 may be within the limits of experimental error. With smooth pipes, however, n may be as low as 1·75, and with ship-shaped bodies of fair form in water is usually about 1·85.

More recent experiments * indicate that no one constant value of n holds over a very wide range of velocities, but that n increases with the velocity, and that a formula of the type

$$R = Av + Bv^2 + Cv^3,$$

where A and B and C are constants depending upon the form and roughness of the body and on the physical properties of the medium, more nearly represents the actual results. Over a moderate range of velocities a single value of n can usually be obtained which gives the resistance, within the errors of observation, and in view of the convenience of such an exponential formula it is commonly adopted in practice.

At velocities above the critical, the direct influence of viscosity increases with the departure of the index n from 2. When $n = 2$ the resistance is proportional to the density of the fluid, and, in similar bodies, to the square of corresponding linear dimensions.

Between the low velocities at which the motion is stream-line, and the high velocities at which it is definitely turbulent, there is a range over which it is extremely unstable, and in which the resistance may be affected considerably by small modifications in the form, presentation, or surface condition of the body. Thus the resistance of a sphere, at a certain velocity whose magnitude depends on the diameter, is actually increased instead of being diminished by reducing its roughness.

In problems occurring in practice, however, velocities are in general well above the critical point. One noteworthy exception is to be found in the flow of oil through pipe lines in which, owing to the high viscosity of the fluid, the motion is usually non-turbulent.

In hydrodynamical problems it is usual to assume that the

* N.P.L., *Collected Researches*, **11**, 1914, p. 307.

esistance depends solely on the relative velocity of fluid and body, nd that it is immaterial whether the body is at rest in a current of uid, or is moving through fluid at rest. Although there is not much irect experimental evidence on this point, it is probable that while ith stream-line motion the resistance is identical in both cases, in ırbulent motion it is not necessarily so, and that it may be sensibly reater when the fluid is in motion than when the body is in motion.

This is to be expected when it is realized that in a fluid in motion ith a mean velocity v, many of its particles have a higher velocity,) that the kinetic energy is greater than that given by the product f the mass and the square of the mean velocity. Any difference ising from this effect will in general only be small, but comparavely large differences may be expected over the range of velocities which the motion is unstable, owing to the fact that, with a staonary body, the interaction occurs in a medium which has an initial ndency to instability owing to its motion.

Thus the system of eddy formation in the rear of any solid body lvancing into still water may reasonably be expected to differ om that behind the same body in a current of the same mean locity, owing to the instability in the latter case of the medium which, and from which in part, it is being maintained.

Except in the case of stream-line flow, the laws of hydronamical resistance can only be deduced experimentally Much formation can, however, be obtained regarding these laws from application of the two allied principles of dynamical similarity id dimensional homogeneity.

Dynamical Similarity

Two systems, involving the motion of fluid relative to geometrilly similar bodies, are said to be dynamically similar when the paths aced out by corresponding particles of the fluid are also geometrically nilar and in the same scale ratio as is involved in the two bodies

The densities of the fluids may be different, as also the velocities ith which corresponding particles describe their paths If the nsities in the two systems are in a constant ratio, and the velocities corresponding particles are also in a constant ratio, then the ratio corresponding forces can be determined. In fact, the scale ratio velocities and that of lengths being both given, the scale ratio of nes is determined, and therefore also the scale ratio of accelerations y means of the fundamental relation, force = mass × acceleration, e ratio of corresponding forces can then be found.

In two systems, denoted by (1) and (2), if w be the weight of unit volume, ρ the density, v the velocity, l any definite linear dimension, and r the radius of curvature of the path, these forces are in the ratio

$$\frac{F_1}{F_2} = \frac{w_1 v_1^2}{r_1} \frac{r_2}{w_2 v_2^2} = \frac{\rho_1 l_1^3 r_2}{\rho_2 l_2^3 r_1} \left(\frac{v_1}{v_2}\right)^2$$
$$= \frac{\rho_1}{\rho_2} \frac{l_1^2}{l_2^2} \left(\frac{v_1}{v_2}\right)^2.$$

It follows that for dynamical similarity corresponding velocities must be such as to make the corresponding forces due to each physical factor proportional to $\rho l^2 v^2$. The velocities so related are termed " Corresponding Speeds ".

Where the only physical factor involved is the weight of the fluid, since the force due to this is proportional to ρl^3, the required condition will evidently be satisfied if $\frac{v_1}{v_2} = \sqrt{\frac{l_1}{l_2}}$.

On the other hand, if viscous forces are all important, since the force due to viscosity equals $\mu \frac{dv}{dl}$ per unit area, where μ is the coefficient of viscosity,

$$\frac{F_1}{F_2} = \frac{\mu_1}{\mu_2} \frac{l_1^2}{l_2^2} \frac{v_1}{v_2} \frac{l_2}{l_1}$$
$$= \frac{\mu_1}{\mu_2} \frac{l_1}{l_2} \frac{v_1}{v_2},$$

and for this to equal $\frac{\rho_1}{\rho_2} \frac{l_1^2}{l_2^2} \left(\frac{v_1}{v_2}\right)^2$ it is necessary that

$$\frac{\rho_1 l_1 v_1}{\mu_1} = \frac{\rho_2 l_2 v_2}{\mu_2},$$

or that
$$\frac{v_1}{v_2} = \frac{\mu_1 \rho_2 l_2}{\mu_2 \rho_1 l_1} = \frac{\nu_1 l_2}{\nu_2 l_1},$$

where ν is the " kinematic viscosity " μ/ρ.

Generally speaking, wherever the influence of gravity is involved in the interaction between a solid and a fluid, as is the case where surface waves or surface disturbances are produced, and where the direct influence of viscosity is negligible, corresponding speeds are proportional to the square roots of corresponding linear dimensions; while where gravitational forces are not involved and where the forces are due to viscous resistances, corresponding speeds are

inversely proportional to corresponding linear dimensions, and directly proportional to the kinematic viscosities

The flow of water from a sharp-edged orifice under the action of gravity is an example of the first type of interaction, while the resistance of an air-ship, or of a submarine submerged to such a depth that no surface waves are produced, is representative of the second type.

The resistance of a surface vessel is one of a series of typical cases, of importance in practice, in which both gravity and viscosity are involved, and in which therefore no two corresponding speeds will satisfy all requirements. In other words, the speeds which give geometrically similar wave formations around two similar ships will not give similar stream-lines in those parts of the systems subject to viscous flow If, however, the influence of one of these factors is much greater than that of the other, approximately similar results, which may be of great value in practice, can be obtained by using corresponding speeds chosen with reference to the important factor. Thus in tank experiments on models of floating vessels the corresponding speeds are chosen with reference to the wave and eddy effects, and are proportional to the square root of corresponding linear dimensions. This involves a scale error for which a correction is made as explained on p 213.

Dimensional Homogeneity

In view of the value of the results which may be obtained by the use of the principle of dimensional homogeneity in problems involving fluid resistance, the method of its general application will now be outlined

The principle of dimensional homogeneity states that all the terms of a correct physical equation must have the same dimensions That is, if the numerical value of any one term in the equation depends on the size of one of the fundamental units, every other term must depend upon it in the same way, so that if the size of the unit is changed, every term will be changed in the same ratio, and the equation will still remain valid

The quantities which occur in hydrodynamics may all be defined in terms of three fundamental units The most convenient units are usually those of mass (M), length (L) and time (t)

Example 1.—Suppose some physical relationship to involve only four quantities, say a force R, a length l, a velocity v, and a density ρ.

Let it be assumed provisionally that the relationship is of the form

$$R \propto l^x v^y \rho^z.$$

Expressed dimensionally, this gives

$$MLt^{-2} = L^x . L^y t^{-y} . M^z L^{-3z},$$

and, on equating the indices of like quantities,

$$1 = z, \; 1 = x + y - 3z, \; -2 = -y;$$

whence $$z = 1, \; y = 2, \; x = 2$$

It follows that $R \propto l^2 v^2 \rho$, provided the initial assumption as to the form of R is correct.

It is possible, however, to obtain the result without making this assumption. What we have really proved, in fact, is that the quantity

$$\frac{R}{l^2 v^2 \rho}$$

is dimensionless Also, since it is given that there is a relationship between R, l, v, ρ, the quantity $R/l^2v^2\rho$ is certainly some function of l, v, ρ, say $f(l, v, \rho)$. Now, since the units of l, v, ρ are independent we can, by changing the unit of l, say, change the numerical value of l without changing the numerical values of v or ρ This change does not change the value of $f(l, v, \rho)$, since it does not change the value of the dimensionless number $R/l^2v^2\rho$, to which $f(l, v, \rho)$ is equal. Hence the function f does not involve l, and similarly it does not involve v or ρ; it is therefore a mere numerical constant, so that $R \propto l^2v^2\rho$

Example 2—If more than four quantities of different kinds are involved, for example R, l, v, ρ, μ, where μ is a viscosity (dimensions $ML^{-1}t^{-1}$, p. 28), the assumption $R \propto l^x v^y \rho^z \mu^p$ would not allow us to determine the values of x, y, z, p by considerations of dimensions, since there would be only three equations in four unknowns. It is possible, however, to obtain a new dimensionless number, not involving R, but of the form

$$\frac{\mu}{l^a v^b \rho^c}.$$

Equating the dimensions of μ to those of $l^a v^b \rho^c$, we find

$$1 = c, \; -1 = a + b - 3c, \; -1 = -b;$$

and $$c = 1, \; b = 1, \; a = 1.$$

Thus $\mu/lv\rho$ is dimensionless.

For brevity, we shall denote the two dimensionless numbers now found by K_1, K_2; i.e.

$$\frac{R}{l^2v^2\rho} = K_1, \qquad \frac{\mu}{lv\rho} = K_2. \quad \ldots\ldots\ldots\ldots(1)$$

It will now be proved that if there is a relationship between R, l, v, ρ, μ it can be expressed in the form

$$K_1 = f(K_2), \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where the form of the function f remains undetermined. In fact, since R by assumption is some function of l, v, ρ, μ, and since we can substitute $lv\rho$ K_2 for μ, it follows that $R/l^2v^2\rho$ is some function of l, v, ρ, K_2, or say

$$K_1 = \phi(l, v, \rho, K_2).$$

Then exactly the same argument as that given under Ex. 1 proves that the function ϕ does not involve l or v or ρ, but only K_2, and this is what was to be proved The relation $K_1 = f(K_2)$ may of course be written in other forms, as for example $K_2 = F(K_1)$ or $\psi(K_1, K_2) = 0$.

The general theorem of dimensionless numbers

The general theorem, of which the two preceding results are particular cases, may be stated as follows *

(1) Let it be assumed that n quantities Q_1, Q_2, , Q_n which are involved in some physical phenomenon, are connected by a relation,

$$F(Q_1, Q_2, \ldots, Q_n) = 0, \quad \ldots\ldots\ldots\ldots(3)$$

containing these quantities and nothing else but pure numbers

(2) Let k be the number of fundamental units (L, M, t, . . .) required to specify the units of the Q's

(3) Let $Q_1, Q_2, \ldots, Q_k$ be any k of the Q's that are of independent kinds, no one being derivable from the others, so that these k might, if we so desired, be taken as the fundamental units

(4) Let Q_x be any one of the remaining $n - k$ quantities Q, and let $\frac{Q_x}{Q_1{}^a Q_2{}^b \ldots Q_k{}^p}$, which we denote by K_x, be the dimensionless quantity formed from Q_x and powers of $Q_1, \ldots, Q_k$.

* E Buckingham, *Physical Review*, IV, 1914, p 345, *Phil Mag*, Nov, 1921, 696.

(5) Then the theorem is that the equation

$$F(Q_1, Q_2, \ldots, Q_n) = 0$$

is reducible to the form

$$\phi(K_1, K_2, \ldots, K_{n-k}) = 0, \quad \ldots\ldots\ldots\ldots(4)$$

or, alternatively,

$$K_1 = f(K_2, K_3, \ldots, K_{n-k}) \quad \ldots\ldots\ldots\ldots(5)$$

The proof follows exactly the same lines as in the two particular examples already given

The actual forms of the functions ϕ and f can only be deduced from experiment

In the most general case of a dynamical relationship between any number of quantities, say n, there will be $n - 3$ quantities (K) of zero dimensions, composed of powers of the n quantities, and deduced in the manner explained above The relationship between the n quantities originally considered reduces to a relationship between the $n - 3$ quantities $K_1, K_2, \ldots, K_{n-3}$ Hence if all but one of the K's are known, the last one is determined

Some of the K's may often be written down from inspection Suppose, for example, that a certain phenomenon involves a time t and an acceleration g, in addition to the R, l, v, ρ, μ of Ex 2 above. Then we see at once that the new K's are tv/l and gl/v^2 The relation between the seven quantities is therefore of the form

$$\psi\left(\frac{R}{l^2v^2\rho}, \frac{\mu}{lv\rho}, \frac{tv}{l}, \frac{gl}{v^2}\right) = 0, \quad (6)$$

or, say,

$$R = l^2v^2\rho\phi\left(\frac{\mu}{lv\rho}, \frac{tv}{l}, \frac{gl}{v^2}\right) \quad \ldots\ldots\ldots\ldots(7)$$

It is useful to remark that any product of powers of the K's is dimensionless. Hence if we multiply the second and third of the arguments of ϕ in (7), we get a new dimensionless number tg/v, which may perfectly well replace one of the two, tv/l and gl/v^2, in (6) and (7).

If two or more quantities of the same kind are involved, as, for example, in the case of the resistance of an airship body, where both the length and diameter of the body affect the result, these may be specified by the value of any one, and by the ratios of the others to this one. Thus, in the problem just considered, if besides

ł, v, ρ, μ, g, t and the length l of a body, there are also involved the readth b and the depth d of the body, the solution is

$$R = l^2v^2\rho F\left(\frac{\mu}{lv\rho}, \frac{tg}{v}, \frac{lg}{v^2}, \frac{b}{l}, \frac{d}{l}\right) \quad \ldots\ldots\ldots(8)$$

It is clear from the above examples that the actual arithmetic ivolved in working out an application of the principle of dimensions ; of the simplest possible kind. The real difficulty is in making ire that all the essential quantities concerned in the phenomenon are eing taken into account. If this preliminary condition is not ılfilled, the result obtained will be quite erroneous

Resistance to the Uniform Flow of a Fluid through a Pipe

An examination of the factors involved in the non-accelerated iotion of a fluid through a pipe would indicate that the pressure rop p/l per unit length of the pipe may depend in some way on the diaieter d, the velocity of flow, and the density and viscosity of the fluid ı a liquid where the effect of elasticity is negligible, it is difficult › imagine any other factor likely to affect the pressure drop, except ie roughness of the pipe walls, and, so long as we only consider pipes the same degree of roughness, the general unreduced relationship of the form

$$F(d, v, \rho, \mu, p/l) = 0 \quad \ldots\ldots\ldots(9)$$

ere the number of dimensionless quantities K is $5 - 3 = 2$ ‹actly as in a former example (p 196) we find

$$K_1 = \frac{pd}{l\rho v^2}, \qquad K_2 = \frac{\mu}{dv\rho}.$$

ıere is some advantage in working with ν, the kinematic viscosity, ıich is equal to μ/ρ, rather than with μ itself. Hence $K_2 = \nu/dv$ The reduced relationship may therefore be written

$$\frac{p}{l} = \frac{\rho v^2}{d}\phi\left(\frac{\nu}{dv}\right), \quad \ldots\ldots\ldots\ldots(10)$$

ıere the form of the function ϕ remains to be found from experiınt. Note that the value of the function ϕ for all values of its ;ument can be found by varying *one only* of ν, d, v.

With stream-line flow, experiment shows that $\frac{p}{l}$ is proportional to v. It follows that $\phi\left(\frac{\nu}{dv}\right)$ must equal $\frac{k\nu}{vd}$, and that

$$\frac{p}{l} = \frac{k\mu v}{d^2},$$

where k is a numerical coefficient. This is Poiseuille's expression for the resistance to viscous flow, the coefficient k having the value 32.*

If the flow is turbulent the pressure gradient is approximately proportional to v^n, where n is between 1·75 and 2·0 In this case $\phi\left(\frac{\nu}{dv}\right)$ must be such as to make $\phi\left(\frac{\nu}{dv}\right) = k'\left(\frac{\nu}{dv}\right)^{2-n}$, so that

$$\frac{p}{l} = k'\frac{\rho v^2}{d}\left(\frac{\nu}{dv}\right)^{2-n}$$

$$= k'\frac{\rho v^n \nu^{2-n}}{d^{3-n}}, \qquad (11)$$

or $$h = \frac{k' v^n \nu^{2-n}}{d^{3-n}} l, \qquad (12)$$

where h is the difference of head at two points distant l apart expressed as a length of a column of the fluid This is the Reynolds† formula for pipe flow

If the foregoing assumptions are correct, on plotting observed values of $\frac{pd}{lv^2\rho}$ against simultaneous values of $\frac{\nu}{dv}$, in any series of experiments in which different liquids or pipes of different diameters but equally rough, are used, the points should lie on a single curve

That this is the case for fluids so widely different as air, water, and oil, has been shown by various observers, notably by Stanton and Pannell‡ (see fig 10 at p. 170, where $R = pd/4l$) An example is given in fig. 1, where experimental points for both air and water lie evenly about the two curves shown The agreement in the case of air is only close where the drop in pressure is so small that the effect of the change of density

* Gibson, *Hydraulics and its Applications* (Constable & Co., 1912), p. 69.
† *Scientific Papers*, Osborne Reynolds, Vol. II, p. 97.
‡ *Phil. Trans. Roy. Soc.* A, **214**, 1914, p 199.

is negligible. For large changes of pressure it may be shown* that formula (12) becomes

$$\frac{\delta p}{l} = \frac{k v'_m{}^n \mu^{2-n}}{d^{3-n}} \left[\frac{p_m}{\mathrm{CT}}\right]^{n-1} \quad \ldots\ldots\ldots\ldots(13)$$

Here T is the absolute temperature of the gas, C is the constant obtained from the relationship $pV = CT$,† and v_m and p_m are the mean velocity and pressure in the pipe. The results of experiments on the flow of air through pipes by several experimenters, with dia-

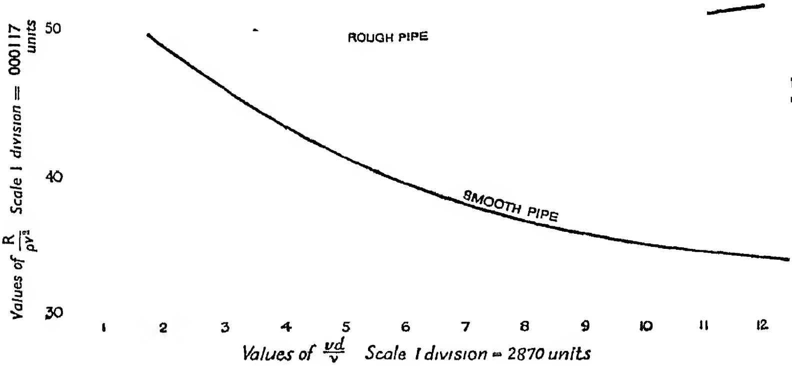


Fig 1

meters ranging from 0·125 in to 0 91 ft., and at velocities from 10 to 40 ft -sec , confirm the accuracy of this formula

Below the critical velocity, $n = 1$, and the formula becomes

$$\frac{\delta p}{l} = \frac{k\mu v_m}{d^2},$$

showing that the pressure drop is now independent of the absolute pressure in the pipe, a result confirmed by experiment.

Equations (10), p. 191, (11), p. 200, show that, with an incompressible fluid, if the resistance to flow varies as v^2, $\phi\left(\frac{\nu}{dv}\right)$ degenerates into a numerical constant. Viscosity ceases to have any direct influence on the resistance,‡ and true similarity of flow should be obtained in all pipes at all velocities. If n is less than 2, the

* Gibson, *Phil Mag*, March, 1909, p 389.

† In the case of air, if the mass is unity, and if p be measured in pounds weight per square foot, the value of C is $53{\cdot}18 \times 32{\cdot}2 = 1710$, while if p be measured in pounds per square inch C becomes 11·9.

‡ See footnote on p. 209

equations show that for similarity of flow in two pipes of differe diameters, or conveying different fluids, it is necessary that $\frac{\nu}{vd}$ shoul be the same in both cases.

This has been shown to be true over a moderate range (diameters by Stanton,* who measured the distribution of velocit with turbulent flow across the diameters of two rough pip of different diameters and repeated the measurements for tw smooth pipes. In the rough pipes ($n = 2$) the velocity distribu tion (CC′, fig. 2) was the same in both pipes and at a

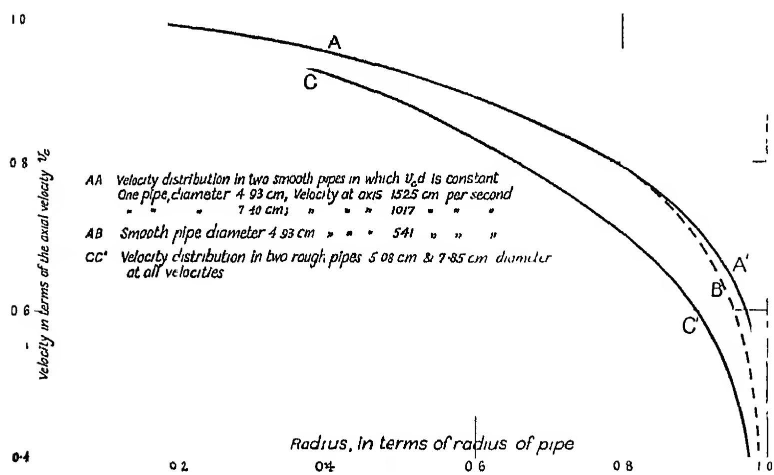


Fig 2

velocities. In the smooth pipes ($n < 2$) identical curves wer obtained only when $\frac{\nu}{vd}$ was the same in each pipe. Under othe conditions the curves were sensibly identical from the centre u to a radius of about 0·8 times the radius of the pipe, but differe appreciably at larger radii (AA′ and AB, fig. 2).

In any type of pipe the "critical velocity" at which the type o motion changes from stream-line to turbulent is obtained with constant value of $\frac{\nu}{vd}$, and this is generally true for fluid motion unde other circumstances.

As already indicated, experiment shows that the resistance t

* *Proc. Roy. Soc.* A, 85, 1911.

flow in a smooth pipe where n is less than 2 is not strictly proportional to any one power of the velocity, and Dr C. H. Lees* has shown that Stanton and Pannell's results for smooth pipes are very closely represented by the empirical relationship

$$\frac{p}{l} = \frac{\rho v^2}{d}\left\{a\left(\frac{\nu}{vd}\right)^{\alpha} + b\right\},$$

where $\alpha = 0{\cdot}35$ and a and b are constants, so that

$$\phi\left(\frac{\nu}{vd}\right) = a\left(\frac{\nu}{vd}\right)^{0\cdot 35} + b.$$

In the rough pipe the ratio of friction to v^2 increases with velocity, and Stanton and Pannell suggest, for both rough and smooth pipes, an expression of the form

$$\mathrm{F} = \rho v^2\left\{\mathrm{A}\frac{\nu}{vd} + \mathrm{K} + \mathrm{B}\frac{vd}{\nu}\right\},$$

in which K depends only on the roughness of the pipe. It will be noted that this relationship is similar to the one obtained by Messrs Bairstow and Booth from experiments on the normal resistance of flat plates (p 213)

Skin Friction

The resistance to the endwise motion of a thin plane through a fluid is usually termed "skin friction" Expressing the resistance by $f\mathrm{A}v^n$, where A is the wetted surface, the values of f for various surfaces in water were determined by Mr Froude † In these experiments a series of flat boards was suspended vertically from a carriage driven at a uniform speed and was towed through the still water in a large basin The boards were $\frac{3}{16}$ in thick, 19 in deep, and varied in length from 1 ft to 50 ft The top edge was submerged to a depth of $1\frac{1}{2}$ in., and the boards were fitted with a cut-water, whose resistance was determined separately

A short résumé of Mr Froude's results is given on the following page, these particular figures referring to a velocity of 10 ft-sec

Here A refers to varnished surfaces or to the painted surfaces of iron ships, B to surfaces coated with paraffin wax, C to surfaces

* *Proc Roy Soc* A, 91, 1914, p 46

† *British Association Report*, 1872.

Length of surface	2 Feet				8 Feet				20 Feet				50 Feet			
Value of n	2·00	1·95	2·16	2·00	1·85	1·94	1·99	2·00	1·85	1·93	1·90	2·00	1·83	—	1·83	2·00
Mean resistance in pounds per square foot	0·41	0·38	0·30	0·90	0·325	0·314	0·278	0·625	0·278	0·271	0·262	0·534	0·250	—	0·246	0·488
Resistance per square foot over last foot, in pounds	0·390	0·370	0·297	0·730	0·264	0·260	0·263	0·488	0·240	0·237	0·244	0·465	0·226	—	0·233	0·456
	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D

oated with tinfoil, and D to surfaces coated with sand of medium oarseness.

The results show that n decreases down to a certain limit, with n increase in length, but is sensibly independent of the velocity. decreases with an increase in length, becoming approximately onstant when the length is large. Owing to viscous drag, those arts of the surface near the prow communicate motion to the vater, so that the relative motion is smaller over the rear part of the urface and its drag per square foot is consequently less, as indicated y a comparison of lines 2 and 3 of the foregoing table

For the case of air, the most reliable work appears to have been lone by Zahm,* who has made observations in an air tunnel 6 ft. quare, on smooth boards ranging from 2 to 16 ft. in length, at elocities from 5 to 25 miles per hour The results are similar to hose obtained by Froude in water, in that the resistance per square oot diminishes with the length, and, for smooth surfaces, varies as $^{1\cdot 85}$. The following results, corresponding to a velocity of 10 t-sec., show that the resistances under similar conditions with hort boards are approximately proportional to the densities of the

Length (feet)	2	4	8	12	16
Mean resistance, pounds per square foot	0·000524	0·000500	0·000475	0·000477	0·000457

wo fluids. Thus for a smooth board (Froude's surface A) the esistance is 790 times as great in water as in air

For strict comparison, experiments carried out with the same alues of $\dfrac{vl}{\nu}$ should be considered. Thus, taking the ratio of ν for ir to ν for water as 13 1, a velocity of 10 ft-sec with the 4-ft board n water would correspond to a velocity of 32·5 ft-sec with the 6-ft board in air Taking the resistance in air as proportional to $^{1\cdot 85}$, the resistance per square foot of the 16-ft. board at this speed is

$$R = 0\cdot 000457 \times (3\cdot 25)^{1\cdot 85} = 0\cdot 00402 \text{ lb},$$

$$\text{and} \qquad R \div v^2 = 0\cdot 00000380$$

he value of $R \div v^2$ for the 4-ft. board in water is 0·00325, the

* *Phil. Mag.*, 8, 1904, pp. 58–66.

ratio of the two being 855, a value only about 4 per cent greater th the relative density of water and air at 60° F.

In view of the fact that one set of experiments was carried o in still water, and the other in an air current whose flow was n perfectly uniform, the agreement between the two sets of resul is very close.

Resistance of Wholly Submerged Bodies

Where a body is submerged in a current to such a depth th no surface waves are formed, gravity has no effect on the resistanc This will happen with a deeply submerged submarine, or wi an air-ship. If the speed is constant, so that there is no acceler tion, and if the liquid is incompressible, or if in a compressib fluid the speeds do not approach the acoustic speed so that pressu changes are so small that the compressibility may be neglecte the resistance R may evidently depend upon the relative veloci of fluid and body, on the density and viscosity of the fluid, and on tl size and shape of the body. In a series of geometrically simil bodies each is defined by a single linear dimension l, and the resi tance R will be given by the relationship

$$F(R, l, v, \rho, \mu) = 0$$

As in the previous examples

$$K_1 = R/l^x v^y \rho^z, \quad K_2 = \mu/l^a v^b \rho^c.$$

Inserting the dimensions of l, v, ρ, R, and μ, and determining th indices x, y, z, a, b, c, so as to make K_1 and K_2 dimensionless, giv the values of K_1 and K_2. These are

$$K_1 = \frac{R}{l^2 v^2 \rho}; \quad K_2 = \frac{l\rho v}{\mu}$$

$$\therefore \quad \psi\left\{\left(\frac{R}{l^2 v^2 \rho}\right), \left(\frac{\mu}{l\rho v}\right)\right\} = 0; \qquad (14)$$

or

$$R = \rho l^2 v^2 \phi\left(\frac{l\rho v}{\mu}\right)$$

$$= \rho l^2 v^2 \phi\left(\frac{lv}{\nu}\right). \qquad (15)$$

The value of the unknown function ϕ might be found by plottin

bserved values of $\frac{R}{l^2v^2\rho}$ against simultaneous values of $\frac{lv}{\nu}$, and y finding an empirical equation to represent the curve joining the lotted points. Moreover, it should be noted that since the values f both terms in the function are dependent on v, the form of the unction, for any liquid, can be determined from experiments on single body at different speeds in the same medium.

In the case of model experiments, if the medium is the same or model and original, and if the suffix m denotes the model, then, f the speeds be chosen so that $v_m = \frac{vl}{l_m}$, we shall have

$$\frac{lv}{\nu} = \frac{l_m v_m}{\nu},$$

o that $\phi\left(\frac{lv}{\nu}\right)$ becomes a constant, and

$$\frac{R}{R_m} = \frac{l^2}{l_m{}^2}\frac{v^2}{v_m{}^2} = 1.$$

The speeds thus related are " corresponding speeds ", and at these peeds the model and its original are " dynamically similar " In his case the corresponding speeds are inversely proportional to the inear dimensions, and at these speeds the resistance of the model nd of the original are equal

Unfortunately this relationship would involve such high speeds n the case of the model as to be of no practical value. If, however, he model experiments can be carried out in a medium whose kinematic viscosity is less than that of the original, the corresponding peeds are reduced Thus by using compressed air in a wind tunnel he corresponding speed is reduced in the same ratio as the density s increased, since the kinematic viscosity of air varies inversely as ts density. Such a wind tunnel is in operation at Langley Field U S A)

Adopting a working pressure of 20 atmospheres

$$\frac{v_m}{v} = \frac{1}{20}\frac{l}{l_m},$$

o that with a 1/10 scale model, the corresponding speed in the vind tunnel would be one-half that of the original, and at these peeds

$$\frac{R_m}{R} = \frac{\rho_n l^2{}_m v^2{}_m}{\rho l^2 v^2} = \frac{1}{20}.$$

With bodies whose resistance is sensibly proportional to th square of the velocity, the form of the function ϕ must be such a to make $\phi\left(\frac{lv}{\nu}\right)$ a constant whose value depends only on the shape o the body, and the resistance is given by

$$R = k\rho l^2 v^2.$$

It is now immaterial at what speed the model experiments are carrie out, so long as this is above the " critical speed ".

This discussion applies equally well to any case of motion of totally immersed body in a medium whose compressibility may b neglected.

Resistance of Partially Submerged Bodies

When a body is partially submerged, or, although submerged is so near the surface that surface waves are produced, part of th resistance to motion is due to this wave formation. The influenc of gravity must now be taken into account, and we have the relation ship

$$F(R, l, v, \rho, \mu, g) = 0.$$

Since there are now 6 quantities involved, $6 - 3\ (= 3)$ K's are required. Taking l, v, and ρ as convenient independent quan tities and proceeding as before,

$$K_1 = R/l^x v^y \rho^z,$$
$$K_2 = \mu/l^a v^b \rho^c,$$
$$K_3 = g/l^\alpha v^\beta \rho^\gamma$$

Inserting the dimensions of R, l, v, ρ, μ, and g, and determining the indices x, a, α, &c , necessary to make K_1, K_2, K_3 dimensionless gives the values of K_1, K_2, and K_3. These are

$$K_1 = \frac{R}{l^2 v^2 \rho}; \quad K_2 = \frac{\mu}{lv\rho}; \quad K_3 = \frac{gl}{v^2};$$

so that $$\psi\left\{\left(\frac{R}{l^2 v^2 \rho}\right), \left(\frac{\mu}{lv\rho}\right), \left(\frac{gl}{v^2}\right)\right\} = 0,$$

or $$R = l^2 v^2 \rho\, \phi\left\{\left(\frac{\nu}{lv}\right), \left(\frac{gl}{v^2}\right)\right\}.$$

In the case of model experiments, it is necessary for dynamical similarity that each of the arguments of ϕ shall have the same value for both model and original. But if, as is usual in practice, both are to operate in water, ν is sensibly constant, while g is also constant, so that for exact similarity both lv and v^2/l would require to be constant In other words, neither v nor l can vary It follows that the lines of flow and the wave formation around a ship and its model in the same fluid cannot simultaneously be made dynamically similar It remains to be seen whether any of the arguments involved in the function ϕ may reasonably be neglected so as to give an approximation which is likely to be of use in practice

Experiment shows that the resistance of a ship-shaped body at the speeds usual in practice is proportional to v^n, where n is approximately 1·83, and the nearness of this index to 2·0 indicates that the direct effect of viscosity is small If it be assumed that this influence of viscosity is negligible,* the argument ν/lv may be omitted from the equation, which now becomes,

$$R = l^2 v^2 \rho \, \phi\left(\frac{gl}{v^2}\right).$$

If now $\dfrac{l}{v^2}$ be made the same both for the model and the original,

$$\frac{R}{R_m} = \frac{\rho}{\rho_m}\frac{l^2}{l_m^2}\frac{l}{l_m} = \frac{\rho}{\rho_m}\left(\frac{l}{l_m}\right)^3 = \frac{\rho}{\rho_m}\frac{D}{D_m},$$

where D is the displacement.

In this case the "corresponding speeds" at which wave and eddy formation are the same for ship and model, are given by

$$\frac{v}{v_m} = \sqrt{\frac{l}{l_m}}.$$

Model Experiments on Resistance of Ships

In practice these corresponding speeds are used, but allowance is made for the different effects of viscous resistances in the two cases by the well-known "Froude" method

* This does not involve the assumption that skin friction is unimportant or that viscosity plays no part in the phenomenon It is in effect assuming that skin friction, instead of being proportional to v^n where n is slightly less than 2, is proportional to v^2. In this case the resistance is due to the steady rate of formation of eddies at the surface of the body, and, once these have been formed and have left the immediate vicinity of the body, the rate at which they are damped out by viscosity has no effect on the drag.

In determining the resistance of any proposed ship, a scale model is made, usually of paraffin wax, and is towed through still water, the resistance corresponding to a number of speeds being measured

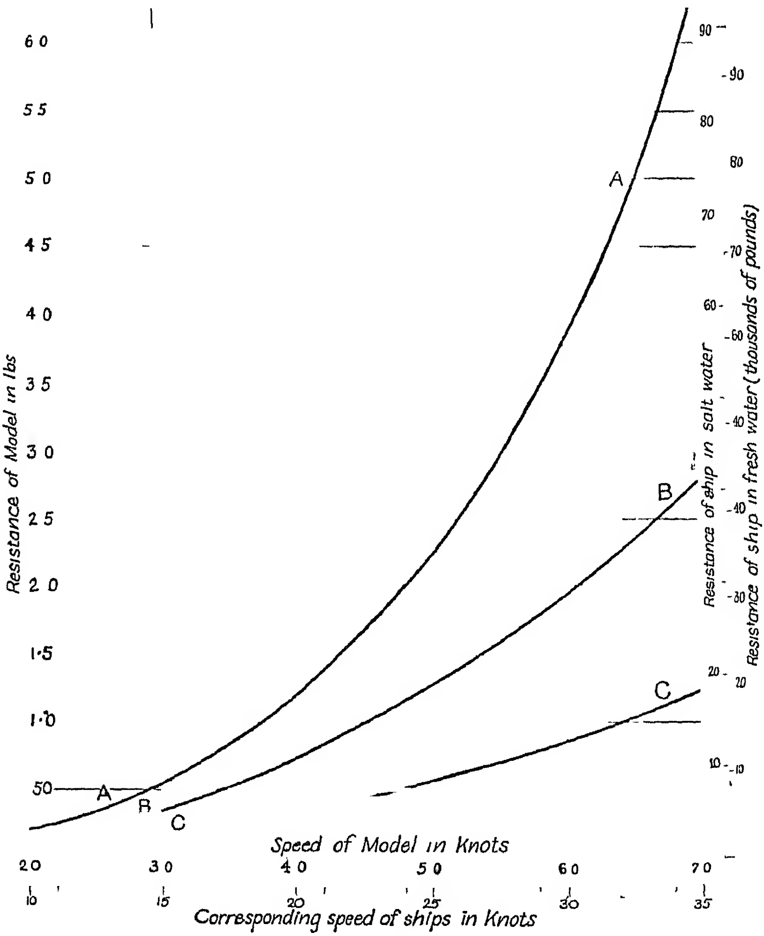


Fig. 3

by dynamometer. A curve AA, fig 3, is plotted showing resistance against speed.

The length and area of the wetted surface being known, the skin friction ($f_m A_m v_m^n$) is calculated, the coefficient being taken from Froude's results on the resistance of flat planes towed endwise.*

* See pp 203, 204

The curve BB of frictional resistance can now be drawn, and the intercept between AA and BB gives the eddy- and wave-making resistance of the model. If now the horizontal scale be increased in the ratio $\sqrt{S:1}$, where S is the scale ratio of ship and model, and if the vertical scale be increased in the ratio D : 1, this intercept gives the eddy- and wave-making resistance of the ship at the corresponding speed. If fresh water is used in the tank, the vertical scale is to be increased again in the ratio of the densities of salt and fresh water The skin friction (fAv^n) of the ship is next calculated and set down as an ordinate from BB to give the curve CC. The intercept between the curves AA and CC now gives, on the large scales, the total resistance of the ship

The resistance at any speed v may be calculated from model observations at the corresponding speed v_m, as follows·

Total resistance of model (observed) .. .. $= R_m$ lb.

Skin friction of model (calculated) .. .. $= f_m A_m v_m^n$ lb.

∴ Wave- + eddy-resistance of model . $= R_m - f_m A_m v_m^n = F$ lb.

∴ Wave- + eddy-resistance of ship in salt water .. . $= DF \times \dfrac{64}{62 \cdot 4}$ lb.

∴ Total resistance of ship .. .. $= \dfrac{64}{62 \cdot 4} DF + fAv^n$ lb.

Scale Effects—Resistance of Plane Surfaces—of Wires and Cylinders—of Strut Sections

From what has already been said, it will be evident that in most model experiments some one of the factors involved tends to prevent exact similarity and introduces some scale effect. It is only when this effect is small, and when its general result is known, that the results of model experiments can be used with confidence to predict the performance of a large-scale prototype.

The resistance of square plates exposed normally to a current,

affords a typical example of scale effects. Expressing the resistance of such plates as

$$R = K'\rho v^2 \text{ in absolute units (British or C.G S)}$$
$$= Kv^2, \text{ where R is in pounds weight per square foot, and } v \text{ is in feet per second,}$$

experiments show the following values of K′ and of K.

	K′	K	Size of Plate.
Dines * ..	·56	00135	1 ft. square
Canovetti † ..	·56	·00134	3 ft diameter (circular)
Eiffel ‡ ..	55	·00133	10 in. square
,, ...	56	00136	14 ,, ,,
,, ...	·59	·00142	20 ,, ,,
,, ...	·61	00147	27 ,, ,,
,, . ..	·62	·00150	39 ,, ,,
Stanton § .	·52	00126	2 ,, ,,
,, .. .	·62	00148	5 ft square
,, ...	·62	·00149	10 ft square

Such experiments show that while $\dfrac{R}{l^2v^2\rho}$ is almost independent of v, it increases by about 18 per cent as the size of the plate is increased from 2 in. to 5 ft. square.

As the compressibility of the air has been neglected in deducing expression (15), p. 206, it is impossible to say without further examination that this effect is not due to compressibility. Indeed if compressibility has any influence on R, a dimensional effect can occur which may be in accordance with a v^2 law of resistance, for, when this is taken into account, (15) becomes

$$R = \rho l^2 v^2 \,\phi\left(\frac{lv}{\nu}, \frac{v}{C}\right), \ldots\ldots\ldots\ldots\ldots(16)$$

where C is the velocity of sound waves in the medium. This may be written

$$R = \rho l^2 v^2 \,\phi\left(\frac{lC}{\nu}, \frac{v}{C}\right),$$

* *Proc. Roy Soc*, 48, p 252.

† Société d'Encouragement pour l'Industrie Nationale, *Bulletin,* 1903, 1, p. 189.

‡ Eiffel, *Résistance de l'Air*

§ N.P.L., *Collected Researches*, 1, p. 261.

and since by hypothesis R is proportional to v^2, this becomes

$$R = \rho l^2 v^2 \psi\left(\frac{lC}{v}\right).$$

An investigation of the possible effect of compressibility shows, however, that this is less than 1 per cent for speeds up to 00 miles per hour, so that an explanation of the observed dimensional effect based on this factor is not admissible.

It has been suggested* that since, as shown by Mr. Hunsaker's observations on circular discs, there is a critical range of speed determined by the form of the edge, and not dependent on the size of plate, the apparent discrepancy between $\frac{R}{v^2}$ = constant and $\frac{R}{l^2}$ = variable may be due to the results of various observers having been affected by such critical phenomena, which were not, however, sufficiently marked to attract attention. To determine whether this explanation is valid would require an experimental investigation of the forms of edge which have been used.

The more probable explanation would appear to be that while experiments on any one plate have been taken as showing the resistance to be proportional to v^2, this is only approximately true.

An examination by Messrs. Bairstow and Booth,† of all reliable experiments on square plates, leads to the conclusion that an empirical formula of the type

$$R = a(vl)^2 + b(vl)^3 \quad \ldots\ldots\ldots\ldots(17)$$

gives a close approximation to the experimental results. For values of vl ranging from 1 to 350, a and b have the values 0·00126 and 0·0000007 respectively Here v is in feet per second and R in pounds

Neglecting compressibility this indicates that $\phi\left(\frac{lv}{\nu}\right)$ of equation (16), p 212, equals $m + nvl$, where m and n are constants for any particular fluid under given pressure and temperature conditions.

It may be noted that experiments by Stanton‡ show that the pressures on the windward side of a square plate are not subject to dimensional effect, but that the whole variation can be traced to changes in the negative pressure behind the plate.

* By Mr. E Buckingham, *Smithsonian Miscellaneous Collections*, **62**, No 4, an., 1916.

† *Technical Report, Advisory Committee for Aeronautics*, 1910–1, p 21.

‡ *Proc. Inst. C. E.*, **171**; also *Collected Researches of the National Physical Laboratory*, **5**, p 192.

Resistance of Smooth Wires and Cylinders

A somewhat similar scale effect is obtained from experiments on the resistance of smooth wires and cylinders. A series of such tests on a range of diameters from 0·002 in. to 1·25 in., with v ranging from 10 to 50 ft.-sec.,* shows that, on plotting $\frac{R}{\rho v^2 d^2}$ against $\frac{vd}{\nu}$ or $\log \frac{vd}{\nu}$, a narrow band of points is obtained which in-

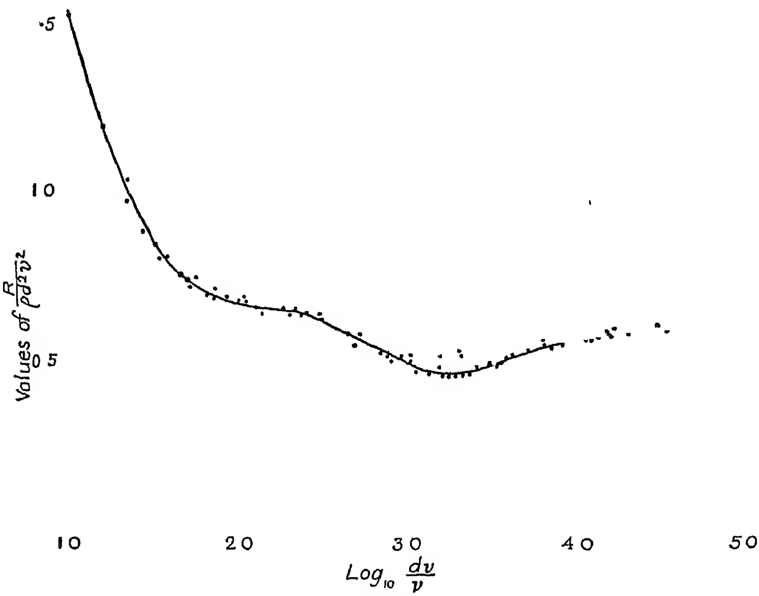


Fig 4

cludes all the experimental results (fig. 4). This shows that for a given value of $\frac{vd}{\nu}$ the value of $\frac{R}{\rho d^2 v^2}$ is the same for all values of v and of d. From this it appears that whereas experiments on a single wire or cylinder indicate that R is nearly proportional to v^2, true similarity of flow is only obtained when $\frac{vd}{\nu}$ is constant.

It becomes very necessary to satisfy this condition with low values of $\frac{vd}{\nu}$, owing to the changes which may occur in the type

* *Reports and Memoranda, Advisory Com. for Aeronautics,* No. 40, March, 1911; No. 74, March, 1913, No 102.

of flow around such bodies at comparatively low velocities or with small diameters. As in all other cases of flow, as this factor is reduced a critical value is ultimately reached where the type of flow undergoes a definite and rapid change, so that the function ϕ ceases to be even approximately constant.

For a given body in a given medium, this critical value of $\frac{vd}{\nu}$ corresponds to a critical speed which may be calculated from the

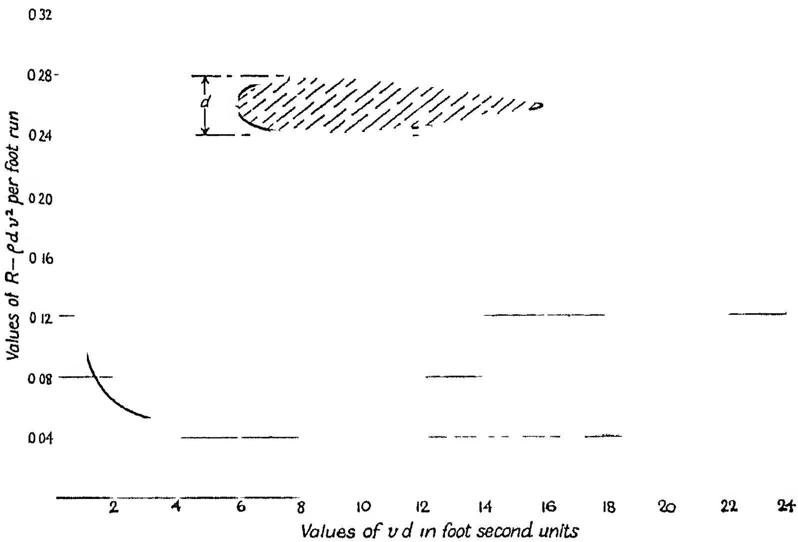


Fig 5 —Resistance of Strut Sections

values of d and ν, if its value has once been experimentally determined for bodies of the given form by varying any one of the variables d, v, and ν.

In some such bodies as spheres and cylinders the law of resistance may change widely with comparatively small alteration in the conditions; thus, for example, at certain speeds the resistance of a sphere may actually be reduced by roughening the surface. In carrying out any such experiments, therefore, it is of the greatest importance that the geometrical similarity between a model and its prototype should be as exact as possible, and that where possible vd should be kept constant.

The variation in the type of flow at a definite critical velocity has been well shown in the case of flow past an inclined plate by

C. G. Eden*. By the aid of colour bands in the case of water, and smoke in the case of air, photographs of the eddy formation in the rear of the plates of different sizes were obtained. These show

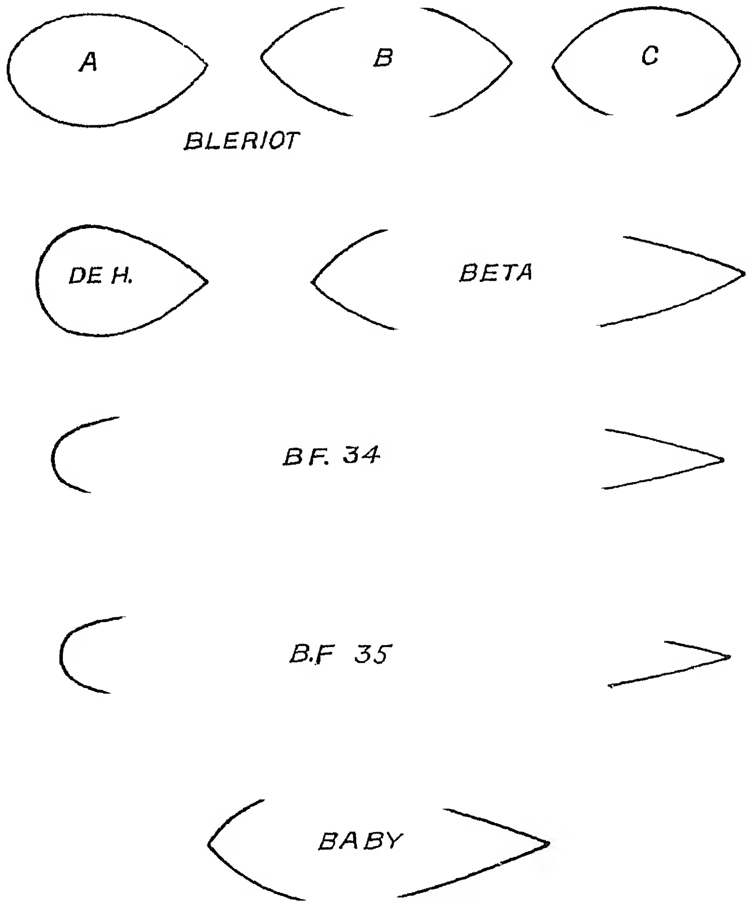


Fig. 6 —Strut Sections.

that the types of flow were similar for both fluids and for all the plates so long as $\frac{vl}{\nu}$ was maintained constant, and that the change over from one type to the other took place at a critical velocity, defined by $v_{\text{crit}} \propto \frac{\nu}{l}$, in each case.

* *Tech. Report of Advisory Committee for Aeronautics*, 1910–1, p. 48; also *R. and M.*, No. 31, March, 1911.

The curve of fig. 5*, p. 215, shows the change in $\frac{R}{\rho d v^2}$ with a variation in vd in the case of a strut of fair stream-line form. Here R is the resistance per foot run of the strut. The curve shows that the resistance is very nearly proportional to v^2 for values of vd greater than 5, but that as vd is reduced below this value the law of resistance suffers a rapid change. Since, below the critical velocity, $R \propto v$, the ordinates of the curve to the left of the critical point will be proportional to $\frac{1}{v}$, and this part of the curve will be hyperbolic.

The following table† shows the resistance of typical strut sections of the types and sizes shown in fig. 6.

Type of Strut	Resistance of 100 Ft of Strut in Pounds at 60 ft.-sec.
Circle, 1 in. diameter	43·0
Ellipse axes, 1 in. × 2 in.	22·2
Ellipse axes, 1 in. × 5 in.	15·2
De Havilland	25·5
Farman	22·9
Bleriot A	23·7
Bleriot B	24·5
Baby	7·9
Beta	6·9
B. F. 34	7·2
B. F. 35	6·3
B. F. 35, tail foremost	10·9

* *Applied Aerodynamics*, Bairstow (Longmans, Green, & Co., 1920), p. 392.
† *Tech. Report of Advisory Committee for Aeronautics*, 1911–2, p. 96.

CHAPTER VI

Phenomena due to the Elasticity of a Fluid

Compressibility

Compressibility is defined (Chapter I, p 16) as the reciprocal of the bulk modulus, i e. by $\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$

The compressibility of water varies with the temperature and the pressure, the values of the bulk modulus, obtained by different observers, being as follows * These values are in pounds per square inch.

Authority	Temperature, Degrees C 0°.	10°	20°	30°	40°	50°	
Landholt and Bornstein	284000	303000	318000	333000	—	—	—
Grassi †	293000	303000	319000	322000	—	—	—
Tait ‡	283000	301000	319000	334000	347000	352000	At low pressures
	292000	311000	328000	340000	347000	347000	At 1 ton per sq in
	300000	321000	332000	346000	339000	339000	At 2 tons per sq in

The bulk modulus K of sea water is about 9 per cent greater than that of fresh water.

* See also Parsons and Cook, *Proc. Roy Soc* A, 85, 1911, p 343 At 4° C. Parsons finds K = 306,000 lb. per square inch at 500 atmospheres, 346,000 lb. at 1000 atmospheres, and 397,000 at 2000 atmospheres. Results of experiments by Hyde, *Proc R S* A, 97, 1920, are in close agreement with these.

† *Annales de Chimie et Physique*, 1851, 31, p 437

‡ *Math. and Phys Papers*, Sir W. Thomson, Vol. III, 1890, p. 517.

For purposes of calculation at temperatures usual in practice, e modulus for fresh water may be taken as 300,000 lb. per square ch, or $43 \cdot 2 \times 10^6$ lb per square foot.

The compressibility is so small that in questions involving water rest or in a state of steady flow it may be assumed to be an incompessible fluid. In certain important phenomena, however, where a idden initiation or stoppage of motion is involved the compressility becomes an important, and often the predominating factor.

In such cases the true criterion of the compressibility or elasticity a fluid is measured by the ratio of its bulk modulus to its density, nce it is this ratio which governs the wave propagation on which ich phenomena depend. In this respect air is only about eighteen mes as compressible as water

For olive oil the value of K at 20° C is 236,000 lb. per square ch (Quincke) and for petroleum at 16·5° C is 214,000 lb per juare inch (Martini). The following are values of K for lubriting oils at 40° C *

Pressure, Tons per square inch	Castor Oil	Sperm Oil	Mobiloil "A"
1	291000	242000	287000
2	302000	252000	291000
5	330000	285000	315000

Sudden Stoppage of Motion—Ideal Case

If a column of liquid, flowing with velocity v along a rigid pipe f uniform diameter and of length l ft., has its motion checked by the istantaneous closure of a rigid valve, the phenomena experienced e due to the elasticity of the column, and are analogous to those btaining in the case of the longitudinal impact of an elastic bar gainst a rigid wall.

At the instant of closure the motion of the layer of water in ontact with the valve becomes zero, and its kinetic energy is conerted into resilience or energy of strain, with a consequent sudden se in pressure. This checks the adjacent layer, with the result iat a state of zero velocity and of high pressure (this at any point

* Hyde, *Proc. R. S.* A, 97, 1920.

being p above the pressure obtaining at that point with stea flow) is propagated as a wave along the pipe with velocity V_p.*

This wave reaches the open end of the pipe after t sec., whe

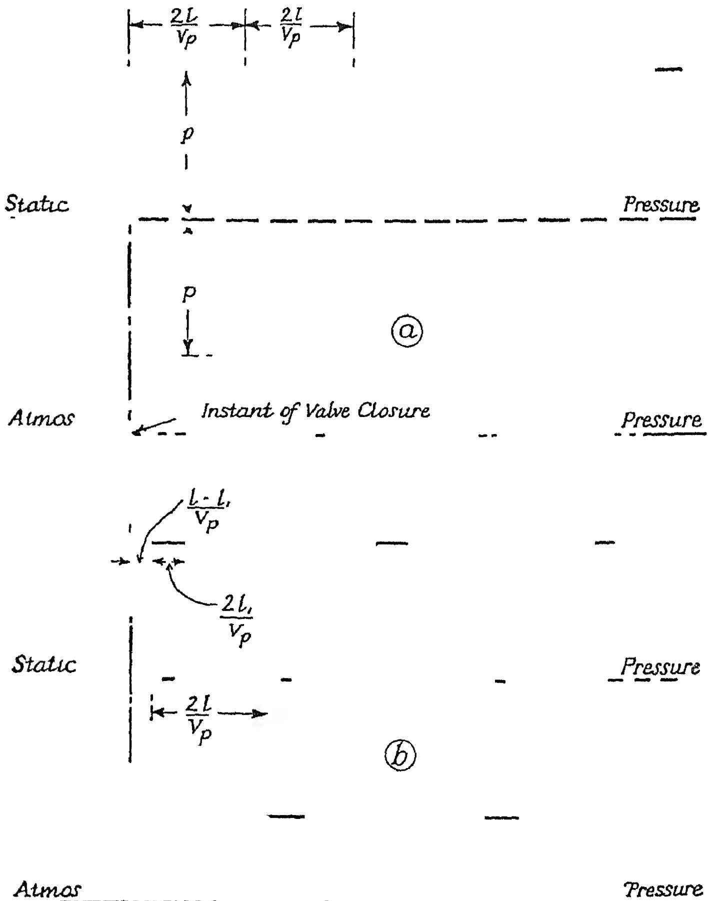


FIG 1 —Ideal Case of Water-hammer in Pipe Line

$t = l \div V_p$. At this instant the column of fluid is at rest and in a state of uniform compression.

This is not a state of equilibrium, since the pressure immediately

* V_p is the velocity of propagation of sound waves in the medium, and is equal to $\sqrt{\frac{Kg}{w}}$, where w is the weight in pounds per cubic foot, and K is in pounds per square foot.

nside the open end of the pipe is p greater than that in the surround-ng medium. In consequence the strain energy of the end layer s reconverted into kinetic energy, its pressure falls to that of the urrounding medium, and it rebounds with its original velocity v owards the open end of the pipe. This relieves the pressure on he adjacent layer, with the result that a state of normal pressure nd of velocity ($-v$) is propagated as a wave towards the valve, eaching it after a second interval $l \div V_p$ sec. At this instant th whole of the column is at normal pressure, and is moving towards he open end with velocity v. The end of the column tends to leave he valve, but cannot do so unless the pressure drops to zero, or so ear to zero that any air in solution is liberated. Its motion is con-equently checked, and its kinetic energy goes to reduce the strain nergy to a value below that corresponding to normal pressure. The pressure d ops suddenly by an amount equal to that through which it originally rose, and a wave of zero velocity and of pressure below normal is transmitted along the pipe, to be reflected from he open end as a wave of normal pressure and velocity towards the alve When this wave reaches the valve $4l - V_p$ sec after the nstant of closure, the conditions are the same as at the beginning of the cycle, and the whole is repeated.

Under such ideal conditions the state of affairs at the valve would e represented by fig 1 *a* At any other point at a distance l_1 from he open end the pressure-time diagram would appear as in fig. 1 *b*

If the velocity were such as to make p greater than the normal bsolute pressure in the pipe, the first reflux wave would tend to educe the pressure below zero. Since this is impossible, the pressure ould only fall to zero, and the subsequent rise in pressure would e correspondingly reduced. Actually, at such low pressures any lissolved air is liberated and the motion rapidly breaks down.

Effect of Friction in the Pipe Line

The effect of friction in the pipe line modifies the phenomena n a complex manner. In the first place the pressure, with steady low, falls uniformly from the open end towards the valve, and the ressure at the valve will be represented by such a line as AB (fig. 2). On closure the pressure here will rise by an amount p as before.

When the adjacent layer is checked its rise in pressure will also e p, but since its original pressure was higher than that at the alve, its new pressure will also be higher. It will therefore tend to

compress that portion of the column ahead of it, and will lose some of its strain energy in so doing. This will result in the pressure at the valve increasing as layer after layer is checked, but since this secondary effect travels back from each layer in turn with a velocity V_p, the full effect at the valve will not be felt until a time $2l \div V_p$ after closure. At this instant the pressure will have risen by an amount which to a first approximation may be taken as $\partial p \div \sqrt{2}$,* where ∂p is the pressure-difference at the ends of the pipe under steady flow.

When reflux takes place the end layer, having transmitted part

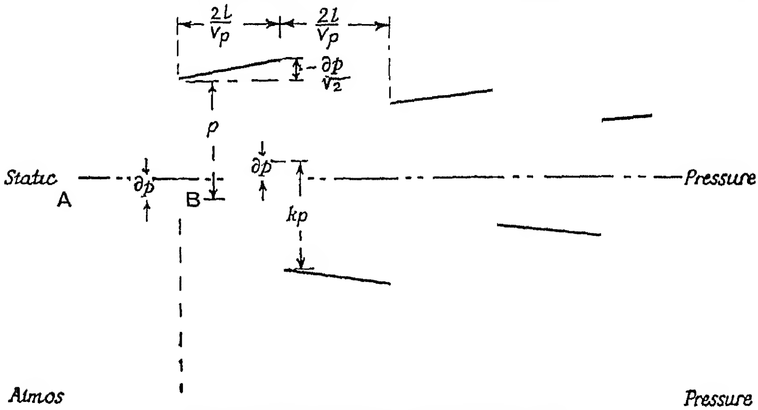


Fig 2 —Water-hammer as Modified by Friction

of its energy along the pipe, can no longer rebound with the original velocity v. Moreover, since at the instant when the disturbance again reaches the valve, and the column is moving towards the open end, frictional losses necessitate the pressure at the valve being ∂p greater than at the open end, the pressure drop will be less than in the ideal case.

* With steady flow the pressure distant x from the valve will be greater than that at the valve by an amount $\partial p \frac{x}{l}$. Therefore excess strain energy of a layer of length δx at this point, due to this pressure $\propto \partial x\left(\frac{\partial p \times x}{l}\right)^2$

Assuming this excess energy to be distributed over the length x between this layer and the valve, it will cause a rise in pressure p', where $(p')^2 x = \partial x\left(\frac{\partial p \times x}{l}\right)^2$.
Integrating to obtain the effect of all such layers from o to x gives

$$(p')^2 = \left(\frac{\partial p}{l}\right)^2 \int_0^x x dx = \left(\frac{\partial p}{l}\right)^2 \frac{x^2}{2}, \text{ and when } x = l,\ p = \frac{\partial p}{\sqrt{2}}$$

This is only a first approximation, since the equalization of pressures will be accompanied by surges which will introduce additional frictional losses.

If the velocity of the first reflux be kv, where k is somewhat less ıan unity, the pressure dıagram will be modified sensibly as ın g. 2.

Friction thus causes the pressure wave to die out rapidly, withıt affecting the periodicity appreciably.

Magnitude of Rise in Pressure, following Sudden Closure

Assuming a rigid pipe, on equating the loss of kinetic energy ər pound of fluıd to the ıncrease in its strain energy or resilıence:

$$\frac{v^2}{2g} = \frac{p^2}{2Kw}.*$$

$$\therefore\ p = v\sqrt{\frac{Kw}{g}} = vV_p\frac{w}{g},$$

here p is the rise in pressure, and v the velocity of flow in feet per ·cond.

Putting $K = 43{\cdot}2 \times 10^6$ lb -sq ft.,
,, $w = 62{\cdot}4$ lb -c ft ,
,, $g = 32\ 2$ ft.-sec.^{2},
thıs becomes $p = 9160v$ lb.-sq ft.
$= 63{\cdot}7v$ lb.-sq ın

Effect of Elasticity of Pipe Line

Owing to the elasticıty of the pipe walls, paıt of the kinetic ıergy of the movıng column ıs expended in stretching these, with resultant increase in the complexity of the phenomena, a reductıon the maximum pressure attained, and an increase ın the rate at hich the pressure waves die out. The state of affairs is then dicated in figs. 3 *a* and *b*, whıch are reproduced from pressurene diagrams taken from a cast-iron pipe line 3·75 ın. dıameter and ;o ft. long †

Fig. 3 *a* was obtained from behind the valve and fig. 3 *b* at a poınt ; ft. from the open end of the pipe.

The elasticity of the pipe line may modıfy the results in two

* If a cube of unıt sıde be subject to a pressure increasıng from o to p, the ange ın volume wıll be p — K, and sınce the mean pressure durıng compres·n ıs $p \div 2$, the work done ıs p^2 — 2K

† Gıbson, *Water Hammer ın Hydraulıc Pıpe Lınes* (Constable & Co., 1908).

ways. If the pipe is free to stretch longitudinally, at the insta of closure the valve end of the pipe and the water column w move together with a common velocity u,* less than v, and a wa of longitudinal extension will travel along the pipe wall. Tl instantaneous rise in pressure at the valve will now be equal to

$$(v - u)\sqrt{\frac{Kw}{g}}.$$

Since the velocity of propagation is much greater in metal tha

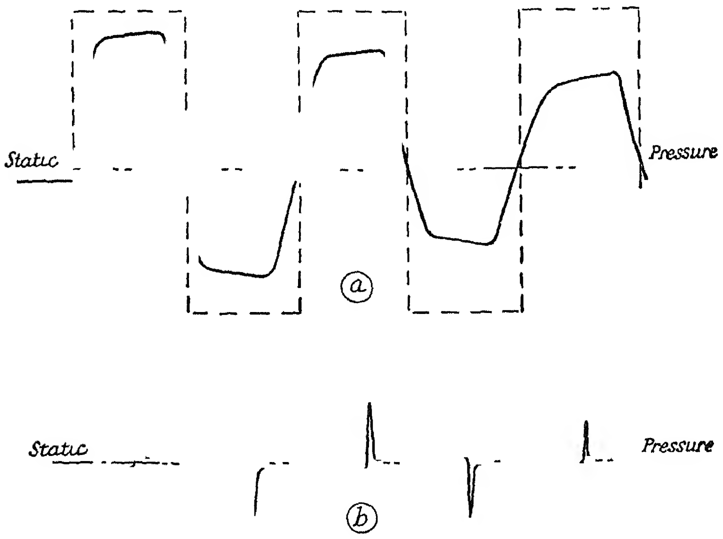


Fig 3—Water-hammer in Experimental Pipe Line

in water, this wave will be reflected from the open end of the pipe and will reach the closed end again before the reflected wave in the water column. At this instant the closed end of the pipe will rebound towards the open end with velocity u, and will produce an auxiliary wave of pressure equal to

$$u\sqrt{\frac{Kw}{g}}$$

* It may readily be shown that $u = v\left\{\frac{1}{1 + \frac{w_m a_m V_m}{w a V_p}}\right\}$, where w_m, a_m, and V_m refer to the weight per cubic foot, the cross-sectional area, and the velocity of propagation in the metal, and w, a, V_p, to the corresponding quantities for water.

in the water column. This will increase the pressure to the value

$$v\sqrt{\frac{Kw}{g}},$$

which would obtain if the pipe were anchored. So that the effect on the maximum pressure attained during this first period $2l \div V_p$ is zero. The effect on the subsequent history of the phenomenon is complex The net effect, however, is to superpose on the normal pressure wave a subsidiary wave of high frequency ($V_m \div 4l$, where V_m is the velocity of propagation of waves in metal) and of magnitude

$$\pm u\sqrt{\frac{Kw}{g}}.$$

If, as is usual in practice, the pipe is anchored so that no appreciable movement of the end is possible, this effect will be small.

The second effect of the elasticity of the pipe line is due to the fact that, since the walls extend both longitudinally and circumferentially under pressure, the apparent diminution of volume of the fluid under a given increment of pressure is greater than in a rigid pipe

The effect of this is to reduce the virtual value of K to a value K′, where *

$$\frac{1}{K'} = \frac{1}{K} + \frac{r}{2tE}\left(5 - \frac{4}{\sigma}\right),$$

where r is the radius of the pipe, t is the thickness of the pipe walls, E is the modulus of elasticity of the material, $1/\sigma$ is Poisson's ratio for the material (approximately 0·28 for iron or steel)

If the pipe is so anchored that all longitudinal extension is prevented, but that circumferential extension is free, this becomes

$$\frac{1}{K'} = \frac{1}{K} + \frac{2r}{tE}.$$

The rise in pressure due to sudden stoppage of motion is now equal to

$$v\sqrt{\frac{K'w}{g}}.$$

* *Hydraulics and its Applications*, Gibson (Constable & Co, 1912), p. 235.

Valve Shut Suddenly but not Instantaneously

If the time of closure t, while being finite, is so small tha

$$t < \frac{l}{V_p} = \frac{x}{V_p},$$

the disturbance initiated at the valve has travelled a distance x and has not arrived at the open end when the valve reaches it seat. In this case, if each part of the column is subject to th same retardation (α), the relationship

force = mass × acceleration

gives $$p = \frac{wax}{g},$$

and since $$\alpha = \frac{v}{t} = \frac{vV_p}{x},$$

this makes $$p = \frac{vV_p w}{g} = v\sqrt{\frac{Kw}{g}} \text{ lb. per square foot,}$$

the value obtained with instantaneous stoppage Whatever the law of valve closure then, if this is completed in a time less than $l \div V_p$ the pressure rise will be the same as if closure were instantaneous

Sudden Stoppage of Motion in a Pipe Line of non-Uniform Section

In such a case the phenomena become very complicatec Let l_1, l_2, l_3, &c., be the lengths of successive sections of a rigi pipe, of areas a_1, a_2, a_3. Following sudden closure of a valve a the extremity of the length l_1, a wave of zero velocity and of pressur $63{\cdot}7\ v_1$ lb. per square inch above normal is transmitted to the junc tion of pipes 1 and 2. Here the pressure changes suddenly to $63{\cdot}7v$ above normal. This is maintained in the second pipe during th passage of the wave, and is followed by a change of pressure to $63{\cdot}7v$ at the junction of 2 and 3, and so on to the end of the line. Bu immediately the pressure at the junction of 1 and 2 attains its valu $63{\cdot}7v_2$, the wave in pipe 1 is reflected back to the valve as a wave c pressure $63{\cdot}7v_2$ and of velocity $v_1 - v_2$, to be reflected from the valve a a wave of zero velocity and pressure $63{\cdot}7\{v_2-(v_1-v_2)\}$ above norma

This wave then travels to and fro along pipe 1, making a complet journey in time $l_1 \div V_p$ sec., until such time as the wave in pipe 2

lected from the junction of 2 and 3 with pressure $63{\cdot}7v_3$ above rmal and with velocity $v_2 - v_3$, again reaches the junction of nd 2 At this instant it takes up a velocity and pressure depending that at the junction end of pipe 1, and as this depends on the ratio the lengths of the branches 1 and 2, it is evident that after the t passage of the wave the phenomenon becomes very involved.

Where a pipe is short the period of the oscillations of pressure any point becomes so small that the pencil of an ordinary indicator unable to record them, and simply records the mean pressure the pipe. Thus where a short branch of small diameter is used the outlet from a long pipe of larger bore, the pressure as recorded an indicator at the valve will be sensibly the same at any instant in the large pipe at the point of attachment of the outlet branch.

Moreover, where a non-uniform pipe contains one section of preciably greater length than the remainder, this will tend to pose its own pressure-change on an indicator placed anywhere the pipe.

These points are illustrated by the following results of experiments by S B Weston * In each case the outlet valve was on the n. length.

Details of Pipe Line	Pressures, Pounds per Square Inch.					
	Calc	Obs	Calc	Obs	Calc	Obs
11 ft of 6-in pipe						
58 ,, 2 ,,	154	73	69	71	—	—
99 ,, 1½ ,,	322	129	143	127	8 9	14·5
4 ,, 1 ,,						
11 ft of 6-in. pipe	1st 1½-in pipe		3-in pipe		2nd 1½-in pipe	
58 ,, 2 ,,						
48 ,, 1½ ,,	71 5	75	18 0	65	71 5	61
3 ,, 3 ,,	142	126	35 5	121	143	114
48 ,, 1½ ,,	180	150	45	150	180	139
4 ,, 1 ,,	268	203	67	207	268	196
82 ft of 6-in pipe	1½-in pipe		2½-in pipe		6-in pipe	
66 ,, 4 ,,						
4 ,, 2½ ,,	120	49	52	22	9 0	4·8
1 ,, 2 ,,	149	62	64 5	36	11 2	6·6
7 ,, 1½ ,,	223	82	97	52	16 7	15 8
6 ,, 1 ,,	466	122	201	99	35 0	36·8

* *Am. Soc. C. E*, 19th Nov, 1884

Sudden Initiation of Motion

If the valve at the lower end of a pipe line be suddenly opened, the pressure behind the valve falls by an amount p lb. per square inch, and a wave of velocity v towards the valve

$$(v = p\sqrt{\frac{g}{Kw}} \text{ approx.})$$

and of pressure p below statical pressure is propagated towards the pipe inlet.

The magnitude of p depends on the speed and amount of opening of the valve, and if the latter could be thrown wide open instantaneously the pressure would fall to that obtaining on the discharge side. In experiments by the writer* with a $2\frac{1}{2}$-in. globe valve on a $3\frac{3}{4}$-in. main 450 ft. long, with the valve thrown open through 0.5 of a complete turn, the drop in pressure was 40 lb per square inch, the statical pressure in the pipe being 45 lb. per square inch, and on the discharge side zero With the valve opened through 0·10 of a turn the drop was 20 lb. per square inch, while with 0·05 of a turn it was 11 lb. per square inch In each case the time of opening was less than 0·13 sec ($l \div V_p$)

With a pipe so situated that the original statical pressure is everywhere greater than p, this pressure wave reaches the pipe inlet with approximately its original amplitude, and at this instant the column is moving towards the valve with velocity v and pressure p below normal.

The pressure surrounding the inlet is however maintained normal so that the wave returns from this end with normal pressure and with velocity $2v$ relative to the pipe. At the valve the wave i reflected, wholly or in part, with a velocity which is the differenc between $2v$ and the velocity of efflux at that instant, and since th velocity of efflux will now be greater than v, the wave velocity wil be less than v, and the rise in pressure less than p above normal This wave is reflected from the inlet to the valve and here the cycl is repeated, the amplitude of the pressure wave diminishing rapidl until steady flow ensues. Fig. 4 shows a diagram obtained unde these circumstances.

Where the gradient of the pipe is such that beyond a certai point in its length the absolute statical pressure is less than th drop in pressure at the valve, the motion becomes partly discontinuou

* Gibson, *Water Hammer in Hydraulic Pipe Lines* (Constable & Co., 1908).

this point on the passage of the first wave, which travels on to the let with gradually diminishing amplitude. The amplitude with hich it reaches the inlet determines the velocity of the reflected

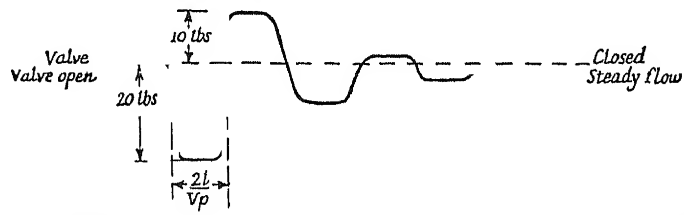


FIG 4.—Diagram of Pressure (per square inch) obtained on Sudden Opening of a Valve

ave, which will be less than in the preceding case, and under such rcumstances the wave motion dies out rapidly.

As the valve opening becomes greater, the efficiency of the valve a reflecting surface becomes less, so that with a moderate opening e pressure may never even attain that due to the statical head

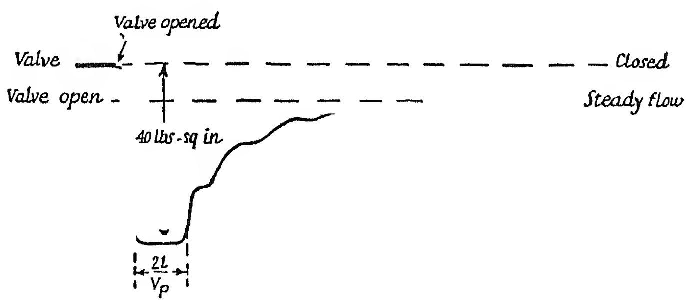


FIG 5 —Sudden Opening of Valve

his is shown in fig. 5, which is a diagram obtained from the experiental pipe line when the valve was opened suddenly $\left(\text{time} < \frac{l}{V_p}\right)$ rough half a turn.

Wave Transmission of Energy

In the systems in common use for the hydraulic transmission energy, water under a pressure of about 1000 lb. per square inch supplied from a pumping-station and is transmitted through pipe ies to the motor. This method involves a continuous flow of the orking fluid, which in effect serves the purpose of a flexible coupling etween the pump and the motor.

It is, however, possible to supply energy to a column of flui enclosed in a pipe line, to transmit this in the form of longitudina vibrations through the column, and to utilize it to perform mechanic work at some remote point. Such transmission is possible in virtu of the elasticity of the column.*

If one end of a closed pipe line full of water under a mean pres sure p_m be fitted with a reciprocating plunger, a wave of alternat compression and rarefaction is produced, which is propagated alon the pipe with velocity V_p. If the plunger has simple harmoni motion, the state of affairs in a pipe line so long that, at the give instant, the disturbance has not had time to be reflected from it further end, is represented in fig 6 The pressure at each poir

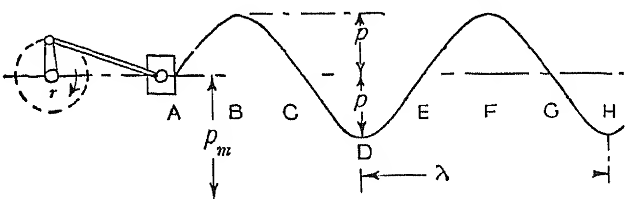


Fig 6

will oscillate between the values $p_m \pm p$, and the velocity betwee the values $\pm v$, where v is the maximum velocity of the pistoı At the instant in question, particles at A, C, E, and G are oscillatir to and fro along the axis of the pipe through a distance r on eac side of their mean position, while particles at B, D, F, and H aı at rest. If n be the number of revolutions of the crank per secon the wave-length $\lambda = V_p \div n$ ft

In a pipe closed at both ends such a state of vibration is reflect from end to end, forming a series of waves of pressure and veloci whose distribution, at any instant, depends on the ratio of the leng l of the pipe to λ.

In the cases where l is respectively equal to $\lambda/4$, $\lambda/2$, and stationary waves are produced as indicated in fig 7.† The exce

* A number of applications of this method have been patented by Mr. Constantinesco.

† The pressure and velocity oscillate in a "stationary" manner, i e there definite points called "nodes" where there is no change in pressure and likew points where the water does not move See any textbook on Sound, e.g. Datt *Sound* (Blackie), p 63, Watson's *Physics*, Poynting and Thomson's *Sound*, & where the subject is fully explained for sound waves.

This distribution, where $l = \lambda/4$, is only possible where oscillation at the e A is possible, as where the pipe is fitted with a free plunger.

ressure at a given point oscillates between equal positive and egative values, the range of pressure being given by the intercept etween the two curves. The velocity at the points of maximum nd minimum pressure, as at A, D, and B in fig. 7 *c*, is zero, while t the points C and E, where the variation of pressure is zero, the elocity varies from $+v$ to $-v$.

In the case where $l = \lambda/4$, the plunger, if free, would continue

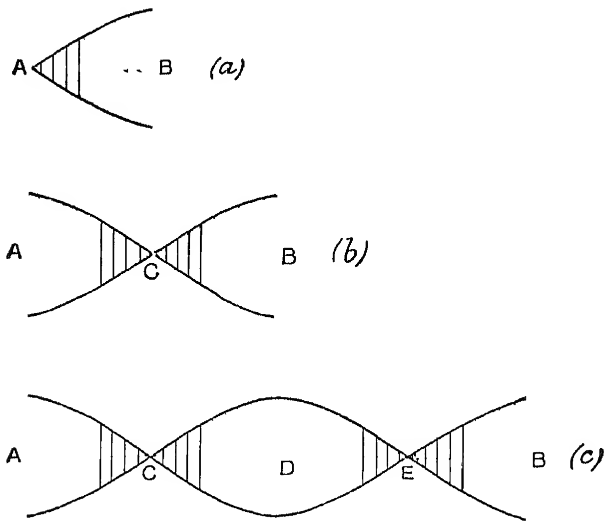


Fig 7—Stationary Waves in a Closed Pipe

) oscillate in contact with the end of the column without the appliation of any external force.

In a pipe fitted with a reciprocating plunger at one end and losed at the other, the wave system initiated by the plunger will be iperposed on this reflected system. Thus if $l = \lambda/2$ or λ, the 'ave initiated by the plunger will be reflected, and will reach the lunger as a zone of maximum pressure at the instant the latter is ompleting its in-stroke and is producing a new state of maximum ressure. The pressure due to the reflected wave is superposed n that due to the direct compression, with the result that the presire is doubled. The next revolution will again increase the presire, and so on until the pipe either bursts or until the rate of dispation of energy due to friction, and to the imperfect elasticity of ie pipe walls, becomes equal to the rate of input of energy by the lunger.

On the other hand, if the length of the pipe be any odd multiple of $\lambda/4$, the pressure at the plunger at any instant, due to the reflected wave, will be equal in magnitude but opposite in sign to that primarily due to the displacement of the plunger, and the pressure on the latter will be constant and equal to the mean pressure in the pipe Except for the effect of losses in the pipe walls and in the fluic column, reciprocation may now be maintained indefinitely withou the expenditure of any further energy. For any intermediate lengths of pipe, the conditions will also be intermediate and the wave distribution complex.

Instead of closing the end of the pipe at B (fig. 8), suppose a piston to be fitted to a crank rotating at the same angular velocity in the same direction, and in the same phase as the crank at A If the column were continued beyond B, the movement of the

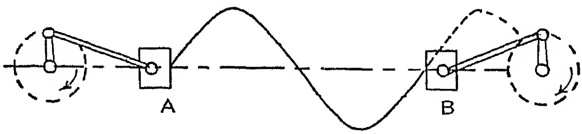


Fig 8

piston would evidently produce in the column a series of wave forming an exact continuation of the wave system between A and B. There will now be no reflection from the surface of the piston and if the latter drives its crank and if the resistance is, at ever instant, equal to the force exerted on the piston by the wave system it will take up the whole energy of the waves produced by piston A It is to be noted that the piston B may be placed at any point in th pipe so long as its phase is the same as that of the liquid at the poin of connection.

If more energy is put in by the piston A than is absorbed by B reflected waves will be formed, and the continuation of the motio will accumulate energy in the system, increasing the maximum pressure until, as in the case of the closed pipe, the pipe will burst

This may be avoided by fitting a closed vessel, filled with liquic having a volume large in comparison with the displacement of th piston, in communication with the pipe near to the piston. Alternatively this may be replaced by a spring-loaded plunger. In eithe case the contrivance acts as a reservoir of energy. If the piston is not absorbing the whole of the energy supplied from A, the liqui in this chamber is compressed on each instroke of the piston A

re-expand on the outstroke, and by giving to it a suitable volume, e maximum pressures, even when the piston B is stationary, ay be reduced to any requiied limits If perfectly elastic, the servoir will return as much energy during expansion as it absorbed iring compression, so that the net input to the driving piston is ly equivalent to that absorbed by piston B.

In the case of a pipe (fig. 7 *c*) whose length is one wave-length, d which is provided with branches at C, D, E, and B, respec- ely one-quarter, one-half, thiee-quarteis, and one wave-length om A, if all the branches are closed, stationary waves will be oduced in the pipe as pieviously described.

If now a motor running at the synchronous speed be coupled the branch at D, this will be able to take up all the energy given the column The stationary half-wave between A and D will nish, being ieplaced by a wave of motion, while the stationary ve will still persist between D and B

Since there is no pressuie vaiiation at C and E, motors coupled these points, with the remaining branches closed, can develop energy.

If a motor be connected at any inteimediate point, part of the out of energy can be taken up by the motor The stationary ve will then persist, but be of reduced amplitude between A and motor, the wave motion ovei this region being compounded of s stationary wave and of a travelling wave conveying eneigy

With a motor at the end B of the line, not absoibing all the energy en by the generator A, there is, in the pipe, a system of stationary ves superposed on a system of waves travelling along the pipe, that there will be no point in the pipe at which the variation of ssuie is always zero It follows that under these conditions a tor connected at any point of the pipe will be able to take some rgy and to do useful work

In practice a three-phase system is usually employed, as giving re uniform torque and ease of starting A three-cylindei gener- -, having cranks at 120°, gives vibrations to the fluid in thiee es, which are received by the pistons of a three-cylinder hydraulic tor having the same crank angles The mean pressure within system is maintained by a pump, which returns any fluid leaking t the pistons.

Theory of Wave Transmission of Energy

The simple theory of the process is outlined below, on the assumption that the friction loss due to the oscillation of the column in the pipe is directly proportional to the velocity. Where such a viscous fluid as oil is used this is true, but where water is used it may or may not be true, depending upon the velocities involved. If the resistance is equal to kv^2 per unit length as with turbulent motion, an approximation to the true result may be attained by choosing such a frictional coefficient k' as will make $k'v = kv^2$ at the mean velocity. At velocities below the critical, $k' = \frac{32\mu}{d^2}$ in pounds per square foot of the cross section (Poiseuille) per unit length of the pipe.

Consider the fluid normally in a plane at x, displaced from that plane through a small distance u, so that its velocity $v = \frac{\partial u}{\partial t}$. The difference of pressure on the ends of an element of length δx, due to the variation in compression along the axis of the pipe, is equal to

$$K\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\delta x,$$

and the equation of motion becomes

$$\rho\frac{\partial^2 u}{\partial t^2}\frac{\pi d^2}{4}\delta x = K\frac{\partial^2 u}{\partial x^2}\frac{\pi d^2}{4}\delta x - \frac{32\mu}{d^2}\frac{\pi d^2}{4}\frac{\partial u}{\partial t}\delta x,$$

$$\text{or} \quad \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial t}, \quad \ldots \quad (1)$$

$$\text{where} \quad a = \sqrt{\frac{K}{\rho}} \text{ and } b = \frac{32\mu}{\rho d^2}.$$

If b is small compared with $4\pi n$, where n is the frequency of the vibration, a solution of equation (1) is

$$u = u_0 e^{-\frac{1}{2}\frac{b}{a}x} \sin 2\pi n\left(t - \frac{x}{a}\right), \quad \ldots\ldots\ldots (2)$$

which represents an axial vibration throughout the column, of maximum amplitude u_0 at the end where $x = 0$.

At any other point the maximum amplitude is $u_0 e^{-\frac{1}{2}\frac{b}{a}x}$, grad

ally diminishing along the pipe owing to the friction term represented by the term b.

The excess pressure p, at any instant and at any point, is equal to $-K\frac{du}{dx}$, i.e.

$$p = -Ku_0 e^{-\frac{1}{2}\frac{b}{a}x} \times \frac{2\pi n}{a} \cos 2\pi n\left(t - \frac{x}{a}\right) \text{ approx.},$$

and the maximum excess pressure, $p_{\max}$, at any point,

$$= -\frac{2\pi n K u_0}{a} e^{-\frac{1}{2}\frac{b}{a}x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The velocity of a particle at x is equal to

$$\frac{\partial u}{\partial t} = 2\pi n u_0 e^{-\frac{1}{2}\frac{b}{a}x} \cos 2\pi n\left(t - \frac{x}{a}\right),$$

and the maximum velocity,

$$v_{\max} = 2\pi n u_0 e^{-\frac{1}{2}\frac{b}{a}x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The energy transmitted by the excess pressure—the mean pressure conveys no energy on the average—across a given section of the pipe in time δt is equal to

$$\frac{\pi d^2}{4} p v \delta t = -\frac{\pi d^2}{4} K \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\delta t.$$

The mean rate of transmission of energy per second over each stroke of the plunger is thus given by

$$-\frac{\pi d^2}{4}\frac{K}{\tau}\int_0^\tau \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}dt,$$

where τ is the duration of a stroke, i.e. of a half-cycle.

Writing $2\pi n\left(t - \frac{x}{a}\right) = \alpha$, $dt = \frac{d\alpha}{2\pi n}$, $\tau = \frac{\pi}{2\pi n} = \frac{1}{2n}$,

this becomes
$$-\frac{Kd^2}{4}\int_0^\pi \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}d\alpha$$

$$= \frac{\pi K}{2a}(\pi n d u_0)^2 e^{-\frac{b}{a}x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$= \frac{1}{2} p_{\max}\, v_{\max}\, \frac{\pi d^2}{4} = E,$$

and the horse-power transmitted, if the foot be the unit of length, is obtained by dividing expression (5) by 550.

The loss of energy per unit length of the pipe, due to friction, and converted into heat, is

$$-\frac{\partial E}{\partial x} = \frac{b}{a}E,$$

and the efficiency of transmission through a pipe line of length l is

$$E_l \div E_0 = e^{-\frac{bl}{a}}.$$

It should be noted that in any application of these results, if the calculations are in English units,

$$\rho = \frac{w}{g} = \frac{62 \cdot 4}{32 \cdot 2}$$

for water, while the value of μ is to be taken in pounds per squar foot, and the pipe diameter in feet.

For a more detailed investigation of the theory, which become complex when a complicated pipe system is used, Mr Constantinesco's original papers should be consulted*.

There is an exceedingly close analogy between wave trans mission by Constantinesco's system and alternating-current electr power transmission; in fact, in the "three-pipe system" tl known facts of three-phase electrical engineering can be applie with scarcely any except verbal changes.

* *The Theory of Sonics* (The Proprietors of Patents Controlling Wave Trar mission, 132 Salisbury Square, E.C., 1920)

NOTE.—The foregoing theory of wave transmission is due to H Moss, D.S See also *Proc Inst Mech Eng*, 1923.

CHAPTER VII

The Determination of Stresses by Means of Soap Films

When a straight bar of uniform cross section is twisted by the application of equal and opposite couples applied at its two ends, it twists in such a way that any two sections which are separated by the same distance are rotated relative to one another through the same angle. The angle through which sections separated by a unit length of the bar are twisted relatively to one another is called the "twist", and it will be denoted by the symbol $\mathfrak{T}$ throughout this chapter. If the section is circular, particles of the bar which originally lay in a plane perpendicular to the axis continue to do so after the couple has been applied

The couple is transmitted through the bar by means of the shearing force exerted by each plane section on its neighbour. The shearing stress at any point is, in elastic materials, proportional to the shear strain, or shear In the case where a bar of circular cross section is given a twist $\mathfrak{T}$, the shear evidently increases from zero at the axis to a maximum at the outer surface of the bar; at a distance r from the axis it is in fact $r\mathfrak{T}$ If two series of lines had been drawn on the surface of the untwisted bar so as to be parallel and perpendicular to the axis, these lines would have cut one another at right angles. After the twist, however, these lines cut at an angle which differs from a right angle by the angle $r\mathfrak{T}$, which measures the shear at the point in question. The shearing strain at the surface of any twisted bar can in fact be conceived as the difference between a right angle and the angle between lines of particles which were originally parallel and perpendicular to the axis.

In the case of bars whose sections are not circular, the particles which originally lay in a plane perpendicular to the axis do not continue to do so after the twisting couple has been applied; the cross sections are warped in such a way that the shear is increased

in some parts and decreased in others. In the case of a bar of elliptic section, for instance, the point on the surface of the bar where the shear is a maximum is at the end of the minor axis, while the point where it is a minimum is at the end of the major axis If the sections had remained plane, so that the shear at any point was proportional to the distance of that point from the axis of twist, the reverse would have been the case.

The warping of sections which were originally plane is of fundamental importance in discussing the distribution of stress in bent or twisted bars. It may give rise to very large increases in stress In the case where the section has a sharp internal corner, for instance, it gives rise to a stress there which is, theoretically, infinitely great.

The method which has been used to discuss mathematically the effect of this warping is due to St. Venant.* If co-ordinate axes Ox, Oy be chosen in a plane perpendicular to the axis of the bar, and if ϕ represents the displacement of a particle from this plane owing to the warping which occurs when the bar is twisted, then St. Venant showed that ϕ satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \qquad \ldots (1)$$

and that it must also satisfy the boundary condition

$$\frac{\partial\phi}{\partial n} = y\cos(xn) - x\cos(yn), \qquad (2)$$

where $\frac{\partial\phi}{\partial n}$ represents the rate of change of ϕ in a direction perpendicular to the boundary of the section, and (xn), (yn) represent the angles between the axes of x and y respectively and the normal to the boundary at the point (x, y).

Functions which satisfy equation (1) always occur in pairs. If ψ is the function conjugate to ϕ, i e. the other member of the pair ψ is related to ϕ by the equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}, \qquad \ldots (3)$$

and ψ also satisfies (1). In the case under consideration it turns out that it is simpler to determine ψ and then to deduce ϕ than to attempt to determine ϕ directly. From (1), (2), and (3) it will be seen that to determine ψ it is necessary to find a function ψ which satisfies

* See Love, *Mathematical Theory of Elasticity*, second edition, Chap. XIV.

1) at all points of the cross section, and satisfies the equation

$$\frac{\partial\psi}{\partial s} = y\cos(xn) - x\cos(yn) \quad \ldots\ldots\ldots\ldots(4)$$

t points on the boundary, where $\frac{\partial\psi}{\partial s}$ represents the rate of variation f ψ round the boundary.

Now $\cos(xn) = \frac{\partial y}{\partial s}$ and $\cos(yn) = -\frac{\partial x}{\partial s}$, so that (4) reduces to $\frac{\psi}{s} = \frac{1}{2}\frac{\partial}{\partial s}\left(x^2 + y^2\right)$, that is to say the boundary condition reduces o

$$\psi = \tfrac{1}{2}(x^2 + y^2) + \text{constant.} \quad \ldots\ldots\ldots\ldots(5)$$

'he advantage in using ψ instead of ϕ is that the boundary condition 5) is more easy to satisfy than (2).

The problem of the torsion of the bar of any section is therefore educed to that of finding a function ψ which satisfies $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$ nd (5) There is an alternative, however If a function Ψ be dened by the relation $\Psi = \psi - \frac{1}{2}(x^2 + y^2)$, then Ψ evidently must atisfy the equation

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + 2 = 0, \quad \ldots\ldots\ldots\ldots(6)$$

t all points of the section, and

$$\Psi = \text{constant} \quad \ldots\ldots\ldots\ldots(7)$$

t the boundary

This function Ψ, besides having a very simple boundary condition, as also the advantage that it is simply related to the shear, in fact he shearing strain at any point is proportional to the rate of change 1 Ψ at the point in question in the direction in which it is a maximum.

Prandtl's Analogy

It has only been possible to obtain mathematical expressions for , ψ, and Ψ in very few cases The stresses in bars whose sections re rectangles, ellipses, equilateral triangles, and a few other special hapes have been found, but these special shapes are of little interest o engineers. There is no general way in which the stresses in wisted bars of any section can be reduced to mathematical terms

The usefulness of equations (6) and (7) does not cease, however vhen Ψ cannot be represented by a mathematical expression. It has

been pointed out by various writers that certain other physica phenomena can be represented by the same equations In some cases these phenomena can be measured experimentally far more easily than direct measurements of the stresses and strains in a twisted bar can be made. Under these circumstances it may be useful to devise experiments in which these phenomena are measured in such a way that Ψ is evaluated at all points of the section The values thus found for Ψ can then be used to determine the stresses in a twisted bar.

Probably the most useful of these "analogies" is that of Prandtl

Consider the equations which represent the surface of a soap film stretched over a hole in a flat plate of the same size and shape as the cross section of the twisted bar, the film being slightly dis placed from the plane of the plate by a small pressure p.

If γ be the surface tension of the soap solution, the equation o the surface of the film is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{p}{2\gamma} = 0, \qquad \ldots(8)$$

where z is the displacement of the film from the plane of xy and and y are the same co-ordinates as before. Round the boundary i e the edge of the hole, $z = 0$.

It will be seen that if z is measured on such a scale that $\Psi = 4\gamma z/p$ then equations (6) and (8) are identical The boundary condition are also the same. It appears therefore that the value of Ψ, corre sponding with any values of x and y, can be found by measuring th quantities p/γ and z on the soap film.

In other words the soap film is a graphical representation of th function Ψ for the given cross section. Actual values of Ψ can b obtained from it by multiplying the ordinates by $4\gamma/p$

To complete the analogy it is necessary to bring out the due connection between the measurable quantities connected with th film and the elastic properties of the twisted bar

If N is the modulus of rigidity of the material and $\mathfrak{T}$ the twi of the bar, the shear stress at any point of the cross sectio can be found by multiplying the slope of the Ψ surface at th point by N$\mathfrak{T}$, so that, if α is the inclination of the bubble to th plane of the plate, the stress is

$$f_s = \frac{4\gamma}{p} N\mathfrak{T}\alpha \quad \ldots\ldots\ldots\ldots\ldots(9)$$

The torque T_q on the bar is given by

$$T_q = 2N\mathfrak{T}\iint \Psi dx, dy,$$

$$\text{or} \qquad T_q = \frac{8\gamma}{p} N\mathfrak{T}V, \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots (10)$$

where V is the volume enclosed between the film surface and the plane of the plate.

The contour lines of the soap film in planes parallel to the plate correspond to the "lines of shearing stress" in the twisted bar, that is, they run parallel to the direction of the maximum shear stress at every point of the section.

It is evident that the torque and stresses in a twisted bar of any section whatever may be obtained by measuring soap films in these respects.

In order to obtain quantitative results, it is necessary to find the value of $\frac{4\gamma}{p}$ in each experiment. This might be done by measuring γ and p directly, but a much simpler plan is to determine the curvature of a film, made with the same soap solution, stretched over a circular hole and subjected to the same pressure difference, p, between its two surfaces as the test film.

The curvature of the circular film may be measured by observing the maximum inclination of the film to the plane of its boundary

If this angle be called β then

$$\frac{4\gamma}{p} = \frac{h}{\sin\beta}, \quad \ldots \quad \ldots\ldots (11)$$

where h is the radius of the circular boundary

The most convenient way of ensuring that the two films shall be under the same pressure, is to make the circular hole in the same plate as the experimental hole.

It is evident that, since the value of $4\gamma/p$ for two films is the same, we may, by comparing inclinations at any desired points, find the ratio of the stresses at the corresponding points of the cross section of the bar under investigation to the stresses in a circular shaft of radius h under the same twist. Equally, we can find the ratio of the torques on the two bars by comparing the displaced volumes of the soap films. This is, in fact, the form which the investigations usually take.

As a matter of fact, the value of $\frac{4\gamma}{p}$ can be found from the test-film itself by integrating round the boundary, α, its inclination to the plane of the plate If A be the area of the cross section, then the equilibrium of the film requires that

$$\int 2\gamma \sin\alpha ds = pA \dots \quad \dots\dots \dots (12)$$

This equation may be written in the form

$$\frac{4\gamma}{p} = 2 \times \frac{\text{area of cross section}}{(\text{perimeter of cross section}) \times (\text{mean value of } \sin\alpha)} \quad (13)$$

By measuring α all round the boundary the mean value of $\sin\alpha$ can be found, and hence $\frac{4\gamma}{p}$ may be determined This is, however, more laborious in practice than the use of the circular standard

It is evident that if the radius of the circular hole be made equal to the value of $\frac{2A}{P}$, where A is the area and P the perimeter of the test hole, then $\sin\beta$ = mean value of $\sin\alpha$ It is convenient to choose the radius of the circular hole so that it satisfies this condition, in order that the quantities measured on the two films may be of the same order of magnitude.

Experimental Methods

It is seen from the mathematical discussion given above that, in order that full advantage may be taken of the information on torsion which soap films are capable of furnishing, apparatus is required with which three kinds of measurements can be made, namely:

(*a*) Measurements of the inclination of the film to the plane of the plate at any point, for the determination of stresses.

(*b*) Determination of the contour lines of the film.

(*c*) Comparison of the displaced volumes of the test film and circular standard for finding the corresponding torque ratio.

The earliest apparatus designed by Dr. A. A Griffith and G. I. Taylor for making these measurements is shown in fig. 1 (see Plate).* The films are formed on holes cut in flat aluminium plates

* From *Proc. Inst. Mech. Eng.*, 14th December, 1917.

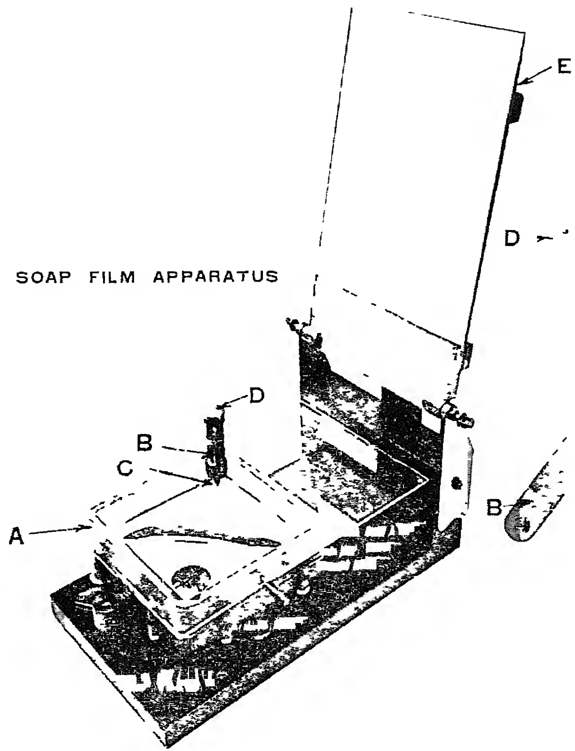


Fig 1

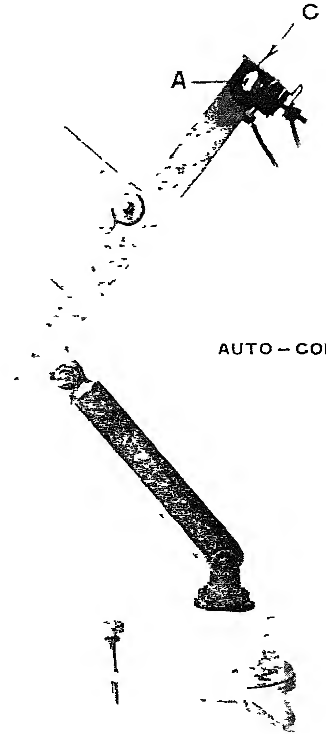


Fig 2(a)

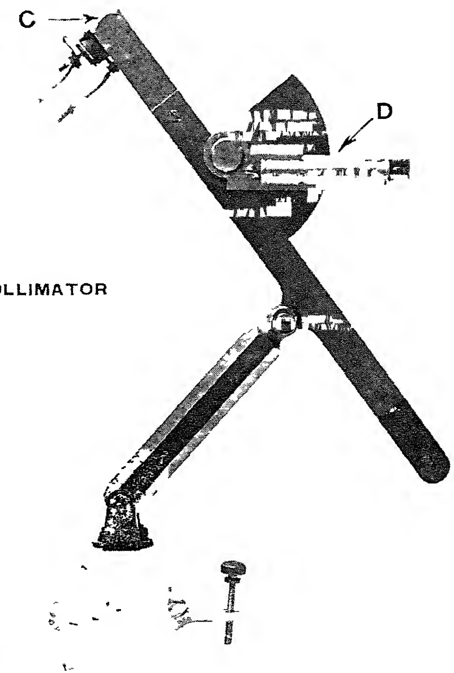


Fig 2(b)

the required shape. These plates are clamped between two lves of the cast-iron box A (fig. 1). The lower part of the box tes the form of a shallow tray $\frac{1}{4}$ in. deep blackened inside and pported on levelling screws, while the upper portion is simply square frame, the upper and lower surfaces of which are machined rallel. Arrangements are made so that air can be blown into the wer part of the box in order to establish a difference in pressure tween the two sides of the film.

In order to map out the contour lines of the film, i.e. lines of ual z, or lines of equal Ψ in the twisted bar, a steel point wetted th soap solution is moved parallel to the plane of the hole till it st touches the film. The point being at a known distance from the ane of the hole marks a point on the film where z has the known lue. The required motion is attained by fixing the point (shown C in fig. 1) to a piece of plate glass which slides on top of the st-iron box. The height of the point C above the plate is adjusted fixing it to a micrometer screw B

In order to record the position of the point C when contact with e film is made, the micrometer carries a recording point D, which ints upwards and is placed exactly over C The record is made a sheet of paper fixed to the board E, which can swing about a rizontal axis at the same height as D To mark any position of e screw it is merely necessary to prick a dot on the paper by inging it down on the recording point The process is repeated a ge number of times, moving the point to different positions on the m but keeping the setting of the micrometer B constant In this ay a contour is pricked out on the paper. To prick out another ntour the setting of B is altered. The photograph shows an actual cord in which four contours traced in this way have been filled with a pencil. The section shown is that of an aeroplane proller blade.

To measure the inclination of the film to the plane of the plate e "auto-collimator" shown in figs 2*a* and 2*b* was devised. Light om a small electric bulb A is reflected from the surface of the m through a V-neck B and a pin-hole eyepiece C placed close the bulb.

Direct light from the bulb was kept away from the eye by small screen. The inclinometer D, which measures the angle ich the line of sight makes with the vertical, consists of a irit level fixed to an arm which moves over a quadrant graduated degrees. In using the auto-collimator the soap-film box is

adjusted till the plane of the hole across which the film is stretched is horizontal.

The volume contained between the film and the plane of the hole can be measured in a variety of ways. One of the most simple is to lay a flat glass plate wetted with soap solution over the test hole in such a way that all the air is expelled from it. The volume contained between the spherical film and plane of the circular hole is then increased by an amount equal to the volume required. This increase in volume can be determined in a variety of ways, one of the simplest being to make measurements with the auto-collimator of the inclination of the spherical film at a point on its edge.

Accuracy of the Method

Strictly speaking, the soap-film surface can only be taken to represent the torsion function if its inclination α is everywhere so small that $\sin\alpha = \tan\alpha$ to the required order of accuracy This would mean, however, that the quantities measured would be so small as to render excessive experimental errors unavoidable A compromise must therefore be effected. In point of fact, it has been found from experiments on sections for which the torsion function can be calculated, that the ratio of the stress at a point in any section to the stress at a point in a circular shaft, whose radius equals the value of $\frac{2A}{P}$ for the section, is given quite satisfactorily by the value of $\frac{\sin\alpha}{\sin\beta}$ where α and β are the respective inclinations of the corresponding films, even when α is as much as $35°$ Similarly, the volume ratio of the films has been found to be a sufficiently good approximation to the corresponding torque ratio, for a like amount of displacement.

In contour mapping, the greatest accuracy is obtained, with the apparatus shown in fig. 1, when β is about 20°. That is to say, the displacement should be rather less than for the other two methods of experiment

In all soap-film measurements the experimental error is naturally greater the smaller the value of $\frac{2A}{P}$. Reliable results cannot be obtained, in general, if $\frac{2A}{P}$ is less than about half an inch, so that a shape such as a rolled I beam section could not be treated satis-

:torily in an appaiatus of convenient size. As a matter of fact, wever, the shape of a symmetrical soap film is unaltered if it be ʼided by a septum or flat plate which passes through an axis of nmetry and is normal to the plane of the boundary. It is theree only necessary to cut half the section in the test-plate and to ıce a normal septum of sheet metal at the lıne of divisıon. An beam, for instance, might be treated by dividing the web at a stance from the flange equal to two or three times the thickness of e web It has been found advisable to carry the septum down rough the hole so that it projects about ⅛ in. below the under side the plate, as otherwise solutıon collects in the corners and spoils e shape of the film.

TABLE I

SHOWING EXPERIMENTAL ERROR IN SOLVING STRESS EQUATIONS BY MEANS OF SOAP FILMS

Section	Radius of Circle	α	β	$\frac{\alpha}{\beta}$	$\frac{\sin\alpha}{\sin\beta}$	True Value	Error, $\frac{\alpha}{\beta}$	Error, $\frac{\sin\alpha}{\sin\beta}$
	In	Deg	Deg				Per Cent	Per Cent
Equilateral triangle height, 3 in	1 00	32 55	21 19	1 536	1 490	1 500	+2 4	−0 7
Square side, 3 in	1 5	29 11	21 34	1 364	1 337	1 350	+1 0	−1 0
Ellipse semi-axes, 2 × 1 in ..	1 296	30 71	24 32	1 263	1 240	1 234	+2 4	+0 5
Ellipse 3 × 1 in	1 410	31 10	24 00	1 296	1 270	1 276	+1 6	−0 5
Ellipse 4 × 0 8 in	1 196	35 35	26 58	1 331	1 293	1 286	+3 5	+0 5
Rectangle 4 × 2 in	1 333	31 70	22 36	1 418	1 380	1 395	+1 6	−1 1
Rectangle 8 × 2 in	1 60	34 83	27 23	1 279	1 247	1 245	+2 7	+0 2
Infinitely long rectangle 1 in wide	1 00	36 42	36 19	1 006	1 005	1 000	+0 6	+0 5

The values set down in Table I indıcate the degree of accuracy ɔtaınable with the auto-collımator ın the determinatıon of the aximum stresses in sectıons for which the toısion functıon ıs known 'hey also give an idea of the sizes of holes whıch have been found ıost convenient in practice. The angles given are α, the maximum ıclination at the edge of the test film, and β, the ınclination at the

* On 4-ın. length.

edge of the circular film of radius $\frac{2A}{P}$. They are usually the means of about five observations and are expressed in decimals of a degree.

The last two columns show the errors due to taking the ratio of angles and the ratio of sines respectively as giving the stress ratio. The error is always positive for α/β, and its mean value is 1·98 per cent. In the case of $\frac{\sin\alpha}{\sin\beta}$ the average error is only 0·62 per cent. In only two instances does the error reach 1 per cent, and in both it is negative. The presence of sharp corners seems to introduce a negative error which is naturally greatest when the corners are nearest to the observation point. Otherwise, there is no evidence that the error depends to any great extent on the shape. Nos. 4, 5, 7, and 8 in the table are examples of the application of the method of normal septa described above in which the film is bounded by a plane perpendicular to the hole at a plane of symmetry.

Table II shows the results of volume determinations made on each of the sections 1 to 8 given in the previous table.

TABLE II

Showing Experimental Error in determining Torques by Means of Soap Films

No.	Section	Maximum Inclination	Observed Volume Ratio	Calculated Torque Ratio	Error
		Deg.			Per cent.
1.	Equilateral triangle height, 3 in.	32·06	1·953	1·985	−1·6
2.	Square side, 3 in.	30·39	1·416	1·432	−1·1
3.	Ellipse semi-axes 2 in. × 1 in.	30·50	1·143	1·133	+0·9
4.	Ellipse: 3 in. × 1 in.	31·01	2·147	2·147	—
5.	Ellipse: 4 in. × 0·8 in.	36·12	3·041	3·020	+0·7
6.	Rectangle: sides, 4 in. × 2 in. ..	31·33	1·456	1·475	−1·3
7.	Reactangle 8 in. × 2 in. ..	35·28	1·749	1·744	+0·3
* 8.	Infinitely long rectangle . ..	36·00	0·858	0·848	+1·2

* On 4-in. length.

The average error is 0 89 per cent. In four of the eight cases con-idered the error is greater than 1 per cent and in three of these it ; negative. One may conclude that the probable error is somewhat reater than it is for the stress measurements, and that it tends to e negative. Its upper limit is probably not much in excess of 2 er cent. The remarks already made regarding the dependence of ccuracy on the shape of the section apply equally to torque measure-nents

When contour lines have been mapped, the torque may be found rom them by integration. If the graphical work is carefully done, he value found in this way is rather more accurate than the one btained by the volumetric method. Contours may also be used o find stresses by differentiation, that is, by measuring the distance part of the neighbouring contour lines; but here the comparison ; decidedly in favour of the direct process, owing to the difficulties nseparable from graphical differentiation. The contour map is, evertheless, a very useful means of showing the general nature of he stress distribution throughout the section in a clear and com-act manner. The highly stressed parts show many lines bunched ogether, while few traverse the regions of low stress, and the direc-ion of the maximum stress is shown by that of the contours at every oint of the section. Furthermore, the map solves the torsion pro-lem, not only for the boundary, but also for every section having he same shape as a contour line

Example of the Uses of the Method

The example which follows serves to illustrate the use of the oap-film apparatus in solving typical problems in engineering lesign.

It is well known that the stress at a sharp internal corner of a wisted bar is infinite or, rather, would be infinite if the elastic equa-ions did not cease to hold when the stress becomes very high. If he internal corner is rounded off the stress is reduced; but so far o method has been devised by which the amount of reduction in train due to a given amount of rounding can be estimated. This roblem has been solved by the use of soap films.

An L-shaped hole was cut in a plate. Its arms were 5 in. long y 1 in wide, and small pieces of sheet metal were fixed at each end, erpendicular to the shape of the hole, so as to form normal septa. The section was then practically equivalent to an angle with arms of

infinite length. The radius in the internal corner was enlarged step by step, observations of the maximum inclination of the film at the internal corner being taken on each occasion.

The inclination of the film at a point 3·5 in. from the corner was also observed, and was taken to represent the mean boundary stress in the arm, which is the same as the boundary stress at a point far from the corner. The ratio of the maximum stress at the internal corner to the mean stress in the arm was tabulated for each radius on the internal corner.

The results are given in Table III.

TABLE III

SHOWING THE EFFECT OF ROUNDING THE INTERNAL CORNER ON THE STRENGTH OF A TWISTED L-SHAPED ANGLE BEAM

Radius of Internal Corner	Ratio $\frac{\text{Maximum Stress}}{\text{Stress in Arm}}$
Inches	
0 10	1 890
0 20	1 540
0 30	1 480
0 40	1·445
0·50	1 430
0·60	1 420
0 70	1·415
0 80	1 416
1 00	1 422
1 50	1 500
2 00	1 660

It will be seen that the maximum stress in the internal corner does not begin to increase to any great extent till the radius of the corner becomes less than one-fifth of the thickness of the arms. A curious point which will be noticed in connection with the table is the minimum value of the ratio of the maximum stress to the stress in the arm, which occurs when the radius of the corner is about 0 7 of the thickness of the arm.

In fig. 3 is shown a diagram representing the appearance of these sections of angle-irons.

No. 1 is the angle-iron for which the radius of the corner is one-tenth of the thickness of the arm. This angle is distinctly weak at the corner.

In No. 2 the radius is one-fifth of the thickness. This angle-iron is nearly as strong as it can be. Very little increase in strength is effected by rounding off the corner more than this. No. 3 is the angle with minimum ratio of stress in corner to stress in arm.

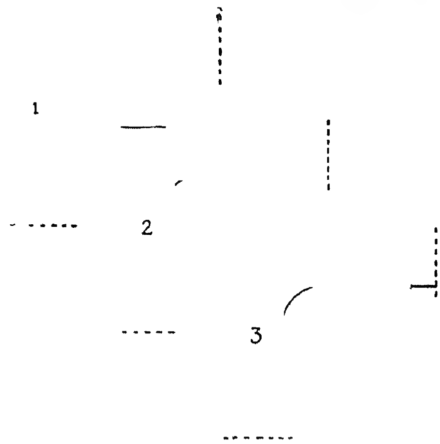


Fig 3

A further experiment was made to determine the extent of the region of high stress in angle-iron No. 1 For this purpose contour lines were mapped, and from these the slope of the bubble was found at a number of points on the line of symmetry of the angle-iron. Hence the stresses at these points were deduced. The results are given in Table IV.

TABLE IV

Showing the Rate of Falling-off of the Stress in the Internal Corner of the Angle-iron

Distance from Boundary	Ratio $\frac{\text{Stress at Point}}{\text{Boundary Stress in Arm}}$
Inches	
0·00	1·89
0·05	1·36
0·10	1·12
0·20	0·77
0·30	0·49
0·40	0·24
0·50	0·00

It will be seen that the stress falls off so rapidly that its maximum value is to all intents and purposes a matter of no importance, if the material is capable of yielding. If the material is brittle and not ductile a crack would, of course, start at the point of maximum stress and penetrate the section.

Comparison of Soap-film Results with those obtained in Direct Torsion Experiments

As an example of the order of accuracy with which the soap-film method can predict the torsional stiffness of bars and girders of types used in engineering, a comparison has been made with the experimental results of Mr. E. G. Ritchie.* The torsional stiffness of any section can be represented by a quantity C such that torque = CN$\mathfrak{T}$, where N is the modulus of rigidity. C has dimensions (length)^4. In Table V column 2 is given the value of C found by soap-film methods, while in column 3 is given the corresponding experimental results taken from Mr. Ritchie's paper.

TABLE V

Section	C (Soap Film)	C (Direct Torsion Experiments)
Angle: 1·175 × 1·175 in.	0·01234 in.^{4}	0·01284 in.^{4}
Angle: 1·00 × 1·00 in.	0·0044 in.^{4}	0·00455 in.^{4}
Tee: 1·58 × 1·58 in.	0·01451 in.^{4}	0·01481 in.^{4}
I-beam: 5·01 × 8·02 in.	1·160 in.^{4}	1·140 in.^{4}
I-beam: 3·01 × 3·00 in.	0·1179 in.^{4}	0·1082 in.^{4}
I-beam: 1·75 × 4·78 in.	0·0702 in.^{4}	0·0635 in.^{4}
Channel: 0·97 × 2·00 in.	0·0175 in.^{4}	0·0139 in.^{4}

Torsion of Hollow Shafts

The method described above must be modified when it is desired to find the torsion function for a hollow shaft. In this case the function satisfies the equation (6) and the boundary conditions are Ψ = constant on each boundary, but the constant is not necessarily the same for each boundary. In order to make use of the soap-film analogy it is therefore necessary to cut a hole in a flat sheet of metal to represent the outer boundary, and to cut a metal plate to represent the inner boundary. These are placed in the correct relative positions in the apparatus shown in fig. 1, and they are set so that they lie in parallel planes. The soap film is then stretched across the gap between them.

The planes containing the two boundaries must be parallel,

*_A Study of the Circular Arc Bow Girder_, by Gibson and Ritchie (Constable & Company, 1914).

ıt they may be at any given distance apart and yet satisfy the con-tıon that $\Psi =$ constant round each boundary. On the other hand ıe contour lines of the film, and hence the value of Ψ, will vary eatly according to what particular distance apart is chosen. The ılution of the torsion problem must be quite definite, so that ust be possible to fix on the particular distance apart at which the anes of the boundaries must be set in order that the soap film retched on them may represent the required torsion function. o do this it is necessary to consider again the function ϕ, which presents the displacement of a particle from its original position ving to the warping of plane cross sections of the twisted material. his function ϕ is evidently a single-valued function of x and y, :. it can have only one value at every point of the material. In neral the values of Ψ found by means of the soap-film apparatus) not correspond with single-valued functions ϕ. On the other hand, ere is one particular distance apart at which the planes of the boun-ıries can be placed so that the Ψ function does correspond with single-valued function ϕ. To solve the torsion problem we must ıd this distance.

If ϕ is single-valued, $\int \frac{\partial\phi}{\partial s}ds = 0$ when the integral is taken round her boundary; and since $\frac{\partial\phi}{\partial s} = -\frac{\partial\psi}{\partial s}$, this condition reduces to $\frac{\partial\psi}{\partial n}ds = 0$. Substituting $\Psi = \psi - \frac{1}{2}(x^2 + y^2)$ and remem-ring that

$$\frac{\partial}{\partial n}(x^2 + y^2) = \frac{\partial y}{\partial n}\frac{\partial}{\partial y}\left(\frac{x^2 + y^2}{2}\right) + \frac{\partial x}{\partial n}\frac{\partial}{\partial x}\left(\frac{x^2 + y^2}{2}\right) = y\frac{\partial x}{\partial s} - x\frac{\partial y}{\partial s},$$

will be seen that

$$\int \frac{\partial\psi}{\partial n}ds = \int \frac{\partial\Psi}{\partial n}ds - 2A, \ldots\ldots\ldots \quad (14)$$

ıere A represents the area of the boundary. The condition that shall be single-valued is therefore

$$\int \frac{\partial\Psi}{\partial n}ds = 2A. \ldots\ldots\ldots\ldots\ldots \quad (15)$$

:ferring again to the soap-film analogy, and putting $\Psi = 4\gamma z/p$, will be seen that (15) is equivalent to

$$2S\int \sin\alpha ds = Ap.. \ldots\ldots\ldots\ldots\ldots (16)$$

Equation (16) applies to either boundary; it may be compared with equation (12), which there applies only to a solid shaft. Taking the case of the inner boundary, it will be noticed that Ap is the total pressure exerted by the air on the flat plate which constitutes the inner boundary. $2\gamma\int \sin\alpha ds$ * on the other hand is the vertical component of the force exerted by the tension of the film on the inner boundary. Hence the condition that ϕ shall be single-valued gives rise to the following possible method of determining the position of the inner boundary. The plate representing it might be attached to one arm of a balance. The film would then be stretched across the space between the boundaries, and if the outer boundary was at a lower level than the inner one the tension in the film would drag the balance down. The pressure of the air under the film would then be raised till the balance was again in equilibrium. The film so produced would satisfy condition (16).

As a matter of fact this method is inconvenient, and another method based on the same theoretical principles is used in practice, but for this and further developments of the method to such questions as the flexure of solid and hollow bars the reader is referred to Mr. Griffith's and Mr. Taylor's papers published in 1916, 1917, and 1918 in the Reports of the Advisory Committee for Aeronautics.

Example of the Application of the Soap-film Method to Hollow Shafts

As an example of the type of research to which the soap-film method can conveniently be applied, a brief description will be given of some work undertaken to determine how to cut a keyway in the hollow propeller shaft of an aeroplane engine, so that its strength may be reduced as little as possible. These shafts used to be cut with sharp re-entrant angles at the bottom of the keyway, and they frequently failed owing to cracks due to torsion which started at the re-entrant corners. It was proposed to mitigate this evil by putting radii or fillets at these corners, and it was required to know what amount of rounding would make the shafts safe.

The shafts investigated were 10 in external and 5·8 in. internal diameter. This was not the size of the actual shafts used in aero-

* The factor 2 comes in owing to the fact that γ is the surface tension of one surface and the film has two surfaces.

ɔlanes, but ıt was found to be the size which gave most accurate esults with the soap-film apparatus.

Some of the results of the experiments are shown graphically in he curve in fig. 4.* In this curve the ordinates represent the maximum

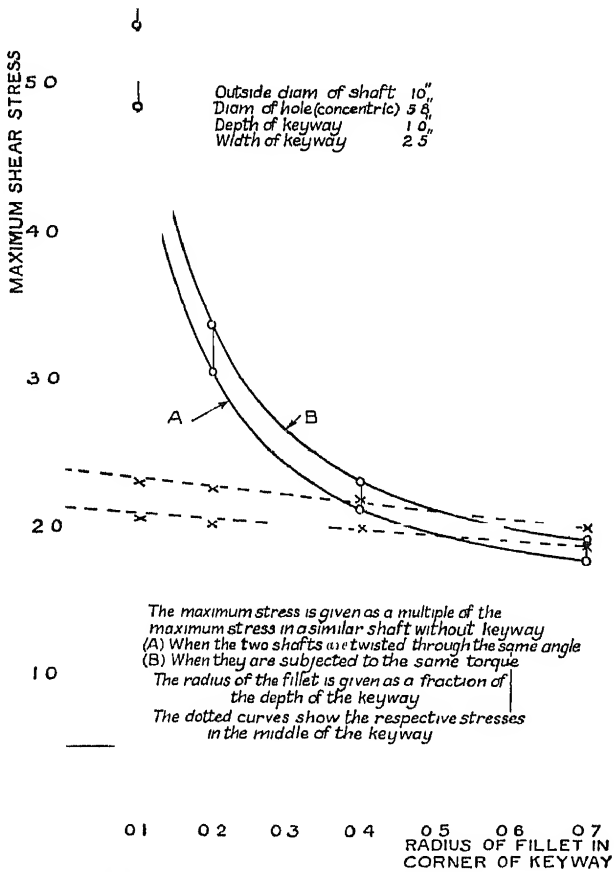


Fig 4 —Torsıonal Strength of Hollow Shaft wıth Keyway

ıhear stress, on an arbitrary scale, while the abscıssæ ıepresent the ·adıus of the fillet, in whıch the internal corners of the keyway were ·ounded off It will be seen that the shaft begins to weaken rapidly vhen the radıus ıs less than about 0·3 ın .

* Thıs diagram and also that shown ın fig 5 aıe taken from Messrs. Grıffith ınd Taylor's report to the Advısory Commıttee for Aeronautics, 1918.

The lines of shearing stress, i.e. the contour lines of the soap film, are shown in fig. 5 for the case when the radius of the fillet is 0.2 in.

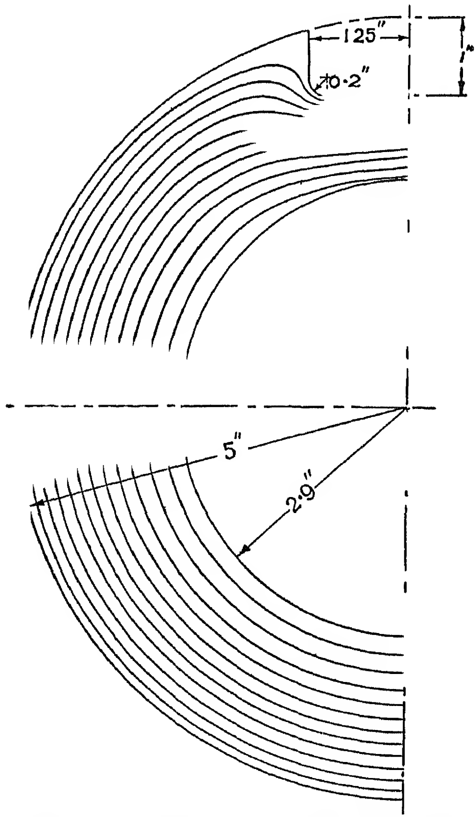


Fig 5 —Lines of Shearing Stress in the Torsion of a Hollow Shaft with Keyway

It will be seen that the lines of shearing stress are crowded together near the rounded corner of the keyway.

CHAPTER VIII

Wind Structure

During the present century great advances have been made in : field of aviation, and problems, some of them entirely new, others der a new guise, have presented themselves. Among the latter ıy be included the problem of *wind structure* Slight changes of : wind, both in direction and in magnitude, are of little account some problems where only the average effect of the wind is of y moment On the other hand, for the aviator these small changes · often of far greater moment than the general drift, especially en his machine is either leaving or approaching the ground. Now s just at this point that the irregularities are often greatest.

Before proceeding to deal with the cause of these various irregu-ities, let us consider what are the governing factors in the move-nt of a mass of air over the surface of the globe.

Apart from the difficulties of dynamics, the general problem is : of much complexity. In the first place, the surface of the earth not at all uniform. It consists of land and water surfaces, and a d surface and a water surface behave quiet differently towards ar radiation, so that air over one area becomes more warmed up n that over another. Further, the land areas are divided into erts and regions rich in vegetation, flat plains, and mountain ranges. ain, water vapour, to whose presence in the atmosphere nearly all teorological phenomena are due, while being added at one place, not subtracted simultaneously at another, so that the amount sent in the atmosphere varies very irregularly. These and other tors tend to render an exact mathematical solution of the problem ctically impossible.

An approximate determination, however, of the effect of the th's rotation on the horizontal distribution of pressure when the moves over the surface of the globe in a simple specified manner, ı be found.

To obtain this approximate solution of the problem, we shall

assume that the air is moving horizontally* with constant linear velocity v, i.e. that a steady state has been reached. The forces acting on a particle of air, in consequence of its motion, under these conditions at a place P in latitude ϕ arise from two causes, (1) the rotation of the earth, and (2) the curvature of the path in which the particle is moving at the instant, relative to the earth. The problem before us therefore is (1) to find the magnitude of the accelerations arising from these causes and (2) to show how the forces required for these accelerations in the steady state are provided by the pressure gradient.

Consider first the effect of the rotation of the earth on great-circle motion. We shall suppose the particle is constrained, by properly adjusted pressure gradients, to move in a great circle through P with uniform velocity. The particle is therefore supposed to move in a path which is rotating in space about an axis passing through its centre.

The rotation of the earth takes places about its axis NS, fig. 1. The great circle Q′PQ is the specified path of the particle. The earth's rotation may be resolved by the parallelogram of rotations into two rotations about any two directions in a plane containing NS Let these two directions be the two perpendicular lines OP and OW′, where O is the centre of the earth and P the point in latitude ϕ referred to above. If the angular velocity of the earth about SN be ω, then the component angular velocities are $\omega\cos\phi$ about OW′ and $\omega\sin\phi$ about OP. As the two axes are mutually perpendicular, it follows that any particle in the neighbourhood of P is in the same relation to OW′ as a particle on the equator is to ON. But a particle on the equator moving with uniform horizontal velocity has an acceleration directed *only perpendicular* to the axis ON, and therefore its horizontal velocity is not affected by the rotation about ON. Similarly the horizontal velocity of a particle near P is affected only by the component $\omega\sin\phi$ about OP, and not by the perpendicular component $\omega\cos\phi$ about OW′. We need consider therefore only the effect of the component $\omega\sin\phi$.

When the particle crosses the point P, it will travel a distance PA $= vdt$ (see fig. 1*a*) in time dt, as the velocity is v. In the same interval of time, the line along which the particle started will have moved into the position PA′, so that the element of arc $ds =$ AA $=$ PA$\omega\sin\phi dt$.

* I e. in a plane perpendicular to the direction of the force compounded of the force of gravity and the centrifugal force.

Also ds or AA′, which is described in a direction perpendicular to PA, may, by the ordinary formula, be expressed in the form

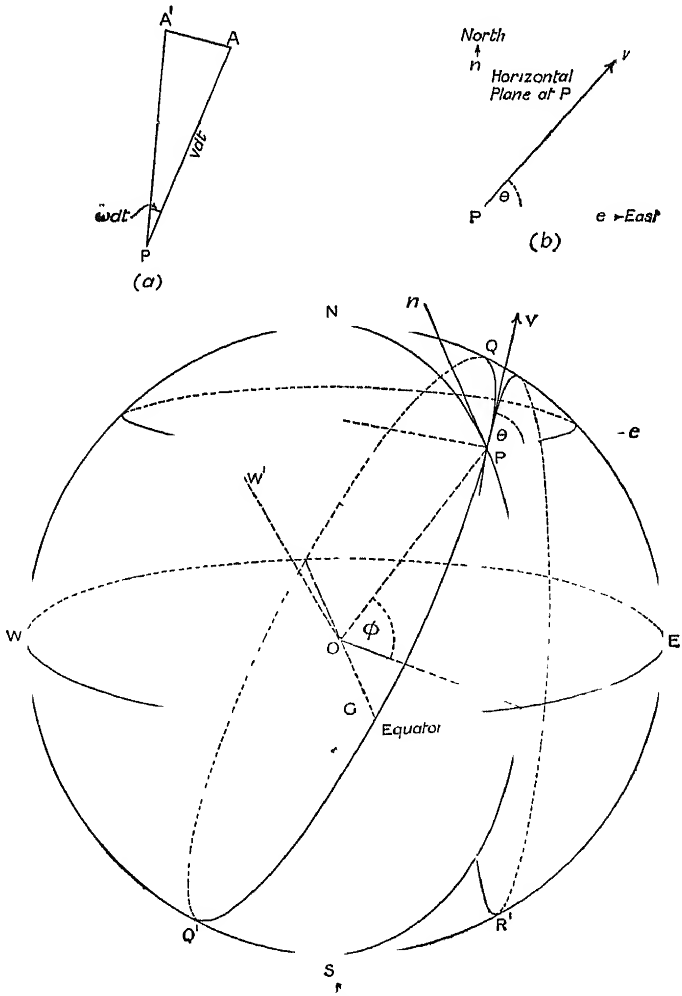


Fig 1

$\frac{1}{2}f(dt)^2$, where f is an acceleration in the direction perpendicular to the direction of motion Hence

$$\tfrac{1}{2}f(dt)^2 = \mathrm{PA}\omega\ \sin\phi dt = v\omega\ \sin\phi(dt)^2,$$

since $\mathrm{PA} = vdt;$

i.e. $f = 2v\omega\ \sin\phi.$

Hence the transverse force (F) necessary to keep a mass (m) of ai moving along a great circle, in spite of the rotation of the earth, is given by

$$F = mf = 2mv\omega \sin\phi, \ldots\ldots\ldots\ldots\ldots (1)$$

acting, in the northern hemisphere, towards the left, in the southern towards the right, when looking along the direction of the wind.

This expression, which is very nearly correct, shows that the deflective force due to the rotation of the earth* on a mass of moving air is (1) directly proportional to the mass, to the horizontal velocity to the earth's angular velocity, and to the sine of the latitude of the place; (2) independent of the direction of the great circle, i.e. of θ (fig. 1); (3) always perpendicular to the instantaneous direction of motion of the air and therefore without influence on the velocity with reference to the surface; (4) opposite to the direction of the earth's rotation.

When the air moves, as specified, in a great circle, the acceleration $2v\omega \sin\phi$ is the only transverse acceleration in the horizontal plane, for the acceleration arising from the curvature of the path relative to the earth (which exists even if ω were zero) is radial and therefore has no appreciable component in the horizontal plane.

Now suppose that the path is not a great circle but a small one, R'P (fig. 1) In addition to the term $2v\omega \sin\phi$ there will now be a term arising from the curvature of the path This term is independent of ω. Let fig. 2 be a section of the sphere through a diameter of the small circle, PR' being the diameter. The path of the air at P is now curved, and if r is the radius of curvature of the path at P, in the horizontal plane, $\frac{v^2}{r}$ is the acceleration in the horizontal plane arising from the curvature of the path. But this acceleration is also the horizontal component of $\frac{v^2}{r'}$ where r' is PM, i.e. the radius of curvature of the small circle. If α is the angular radius of the small circle it is also the inclination of the horizontal plane to the plane of the small circle (see fig. 2), hence

$$\frac{v^2}{r} = \frac{v^2}{r'} \cos\alpha;$$

$$\therefore r' = r \cos\alpha,$$

* I.e the force F, *reversed*.

e. N (fig. 2) is the centre of curvature of the path in the horizontal lane.

It is also clear from fig. 2 that R $\sin\alpha = r'$, hence

$$\frac{v^2}{r} = \frac{v^2}{r'}\cos\alpha = \frac{v^2}{\mathrm{R}\sin\alpha}\cos\alpha = \frac{v^2}{\mathrm{R}}\cot\alpha.$$

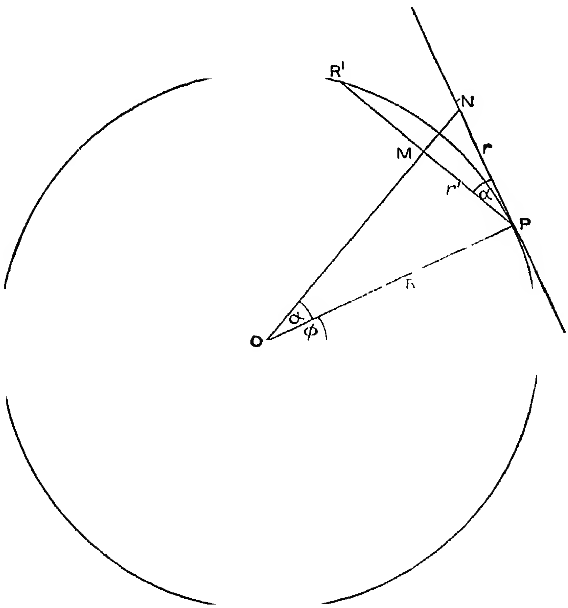


Fig 2 —Relation between Radii of Curvature of the Path on the Earth and the Path in the Horizontal Plane PM = r' PN = $r'/\cos\alpha = r$ PO = R

the air is moving freely in space, i e if the barometric pressure uniform, the resultant horizontal acceleration is zero, i.e.

$$2v\omega\sin\phi + \frac{v^2}{\mathrm{R}}\cot\alpha = 0.$$

[ence " free " motion is only possible when one of these accelera-ons has the opposite direction to the other, and

$$2v\omega\sin\phi = \frac{v^2}{\mathrm{R}}\cot\alpha$$

numerically, i.e. when the acceleration due to path curvature balances that due to the earth's rotation.

When the barometric pressure is not uniform, we proceed thus:

If the particle or element of air at P occupies the volume of a small cylinder, of length δn in the direction of the *outward* drawn normal at P to the path of the air in the horizontal plane, and of unit cross-sectional area, the force on the air in the *inward* direction due to variation of pressure is $\left(\frac{\partial p}{\partial n}\delta n\right)$. The mass of air is $\rho\delta n$ where ρ is the density at P, hence

$$\frac{\partial p}{\partial n}\delta n = \rho\delta n\left[2v\omega\sin\phi + \frac{v^2\cot\alpha}{R}\right].$$

$$\therefore \frac{\partial p}{\partial n} = 2\rho v\omega\sin\phi + \frac{\rho v^2\cot\alpha}{R}. \qquad \text{(2)}$$

The formula is true for positive and negative values of v, remembering that $\frac{\partial p}{\partial n}$ is the gradient of pressure in the outward direction of the normal, and that the rotational term in the acceleration is towards the left hand when looking along the direction of the wind in the northern hemisphere. The two cases of *cyclonic* and *anticyclonic* wind (i.e. $+v$ and $-v$) are shown in fig. 3. The forces indicated in this figure are those required to keep the air in its assumed path, relative to the earth. These forces are provided by the pressure gradient. If we take the *numerical* value of the pressure gradient and the wind speed, then

$$\frac{\partial p}{\partial n} = 2\rho v\omega\sin\phi + \frac{\rho v^2}{R}\cot\alpha \qquad \text{(3)}$$

for the *cyclone*, where $\frac{\partial p}{\partial n}$ is the rate of rise of pressure outwards. For the anticyclone,

$$\frac{\partial p}{\partial n} = 2\rho v\omega\sin\phi - \frac{\rho v^2}{R}\cot\alpha, \qquad \text{(3a)}$$

where $\frac{\partial p}{\partial n}$ is the rate of rise of pressure *inwards*. Both cases are included in (2) without ambiguity.

These expressions give a value of the wind velocity called the 'adient wind velocity. The direction of this gradient wind according › the previous reasoning is *along* the isobars, and is such that to one ıoving with it in the northern hemisphere, the lower pressure is ı the left hand. It must be distinctly understood that in the above ‹pressions for the gradient wind a steady state has been reached; ıd further, it is assumed in arriving at these expressions that there

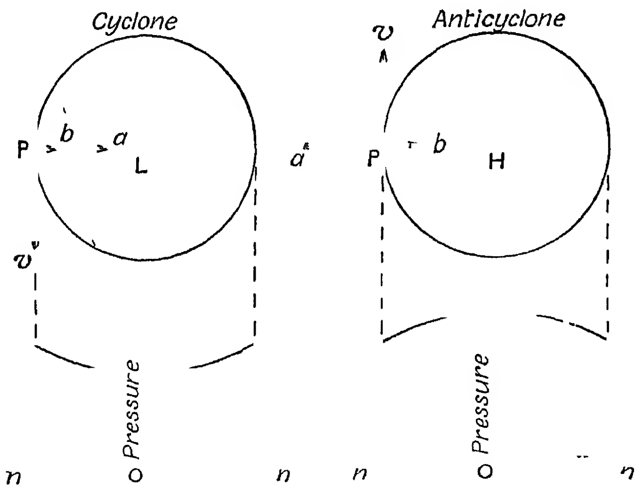


Fig 3

$Pa = 2\rho v\omega \sin\phi$
= term arising from rotation of the earth

$Pb = \frac{\rho v^2 \cot a}{R}$
= term arising from the angular radius of the path

no friction between the air and the surface of the earth over which s passing.

Under actual conditions the relation cannot be satisfied exactly there is always a certain amount of momentum absorbed from the eam of air by the friction at the surface. This absorption of ergy is manifested by the production of waves and similar effects water surfaces, on forests, and on deserts. Yet under the most favourable conditions this relation between wind and pressure can recognized, and therefore it must be an important principle in the ucture of the atmosphere. Also when we ascend into the at- sphere beyond the limits where the influence of surface friction ikely to be felt, we find very little difference in the velocity of the

wind for hours on end. According to Shaw,* "pressure distribution seems to adjust itself to the motion of the air rather than to speed it or stop it. So it will be more profitable to consider the *strophic* balance between the flow of air and the distribution of pressure as an axiom or principle of atmospheric motion." This axiom he has enunciated as follows:† "In the upper layers of the atmosphere the steady horizontal motion of the air at any level is along the horizontal section of the isobaric surface at that level, and the velocity is inversely proportional to the separation of the isobaric lines in the level of the section."

Throughout this short study of wind structure we shall follow Shaw therefore, and regard the wind as balancing the pressure gradient. It may be argued that this assumption strikes at the root of the processes and changes in pressure distribution one may desire to study. The results of investigation appear to indicate, however, that in the free atmosphere, at all events, the balance is sufficiently good under ordinary conditions for us to take the risk and accept the assumption. Under special circumstances and in special localities there may occur singular points where the facts are not in agreement with the assumption, but the amount of light which can be thrown upon many hitherto hidden atmospheric processes, appears to justify our acceptance of it.

In the expression for the calculation of the gradient wind the right-hand side consists of two terms The first term, $2\rho v w \sin\phi$, is due as we have seen to the rotation of the earth, and in consequence has been called the *geostrophic* component of the pressure gradient. The other part, $\frac{v^2\rho}{R}\cot\alpha$, arises from the circulation in the small circle of angular radius α, and so has been termed the *cyclostrophic* component. With decrease in ϕ, i.e. the nearer we approach the equator, α remaining constant, the first component therefore becomes less and less important, the balance being maintained by the second term alone practically. On the other hand, with increase in α, i.e. the nearer we approach to the condition of the air moving in a great circle, ϕ remaining constant, the second term becomes less and less important, until finally with the air moving on a great circle the gradient and the geostrophic wind are one and the same. Consequently in the pressure distributions in mean latitudes where the radius of curvature of the path is

* *Manual of Meteorology*, Part IV, p. 90
† *Proc. Roy Soc. Edin.*, 34, p. 78 (1913).

ıerally very large, the geostrophic wind is commonly taken as : gradient wind.

Having obtained expressions indicating the connection between : pressure gradient and the theoretical velocity of the wind, we ıll now consider some of the reasons for the variations of the wind ocity from this theoretical value.

In the equatıon for the gradient wind and in the statements made ;arding the effects of friction on the wind, there is nothing to lıcate that the flow of air is not steady But it is a perfectly well

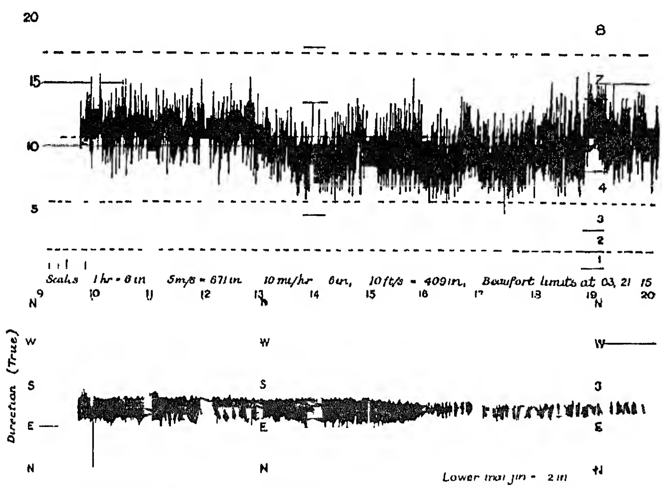


Fig 4 —Anemometer Record at Aberdeen Observatory, 26th September, 1922

ıwn fact that the air, at all events near the surface, does not flow h a constant velocıty even for a very short ınterval of tıme. This ıteadiness in the wınd velocıty is exhıbıted very well by the records ;elf-recordıng anemometers Fig. 4, whıch is part of the record 26th September, 1922, at Aberdeen, exhıbıts this moment-to-ment vaııation. Not only does the velocıty vary but the dırectıon ı shows a simılar varıatıon, as indicated by the lower trace in the ıre. As a rule the greater the variatıon in velocıty, the greater also variatıon ın dırection.

The varıatıons both ın velocity and direction are very largely ıendent upon the nature of the surface over whıch the aır ıs sıng, i.e. the nature of the records ıs greatly affected by the osure of the anemometer. A comparison of figs. 4 and 4*a* reveals ; very plainly The first, as stated above, is a record from the

anemometer at King's College Observatory, Aberdeen. The head of the instrument is 40 ft. above the ground, the instrument itself being housed in a small hut* in the middle of cultivated fields and placed about ½ mile from the sea. The second is a record from an anemometer situated about 5 miles inland from the first, at Parkhill Dyce, and belonging to Dr. J E. Crombie. The exposure in this case is over a plantation of trees, and though the head of the instrument is 75 ft. above the ground, it is only 15 ft above the level of

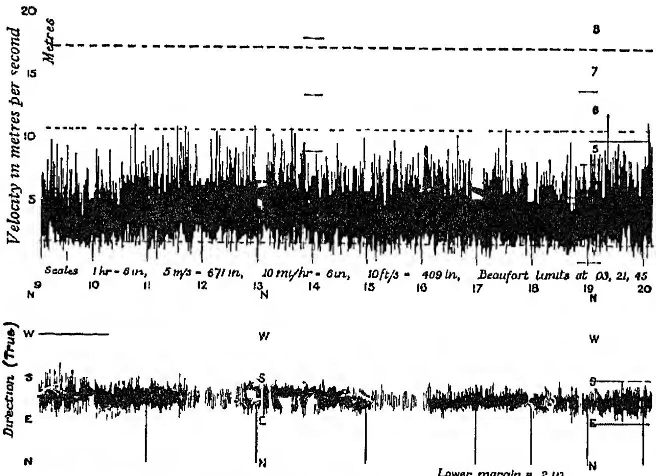


Fig 4a—Anemometer Record at Parkhill, Dyce, 26th September, 1922

the tree-tops. The two records refer to the same day, and the anemometers are situated comparatively close the one to the other, yet the "gustiness" as indicated on the second is much greater than that on the first. At the same time the average velocity of the wind in the second case is considerably reduced by the general effects of the nature of the exposure.

Other factors which affect this variation of the wind are to be found in the undisturbed velocity of the wind in the upper air and in the temperature of the surface of the ground, i.e. in the time of day and in the season of the year. A great deal of light has been thrown on these variations of the wind near the surface by G. I. Taylor through his investigations of eddy motion in the atmosphere.†

*The position of this hut is about a quarter of a mile directly eastwards from the position of the anemometer shown in fig 6 Fig 6 is compiled from records taken in the old position

†"Phenomena connected with Turbulence in the Lower Atmosphere", *Proc. Roy Soc.* A, 94, p 137 (1918).

From the aviator's point of view these variations are often of prime importance. Beginning therefore at the surface, we shall endeavour to ascertain how the actual wind is related to the geostrophic or gradient wind for various exposures, and afterwards determine how these relations alter as we ascend higher into the atmosphere.

When the hourly mean values of the surface wind velocities are examined for any land station, it is found that there is a diurnal

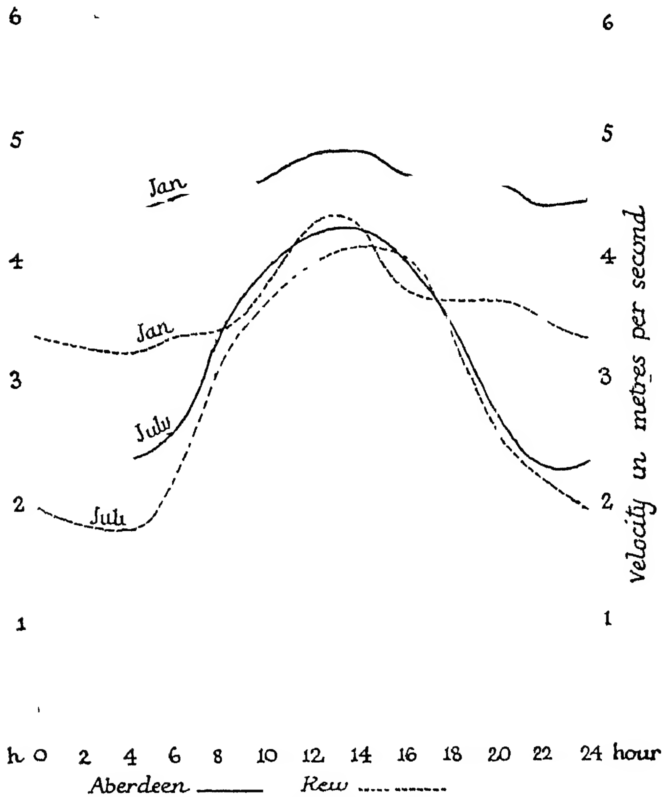


Fig 5—Diurnal Variation in Wind Velocity for January and July, for the period 1881–1910

and also a seasonal variation in the velocities. Fig 5 represents this diurnal variation for the two stations, Aberdeen and Kew, for the months of January and July. On the other hand, no corresponding diurnal variation of the barometric gradient is to be found for these stations. The diurnal variation of the wind is evidently

dependent upon the diurnal and seasonal variations of temperature, and therefore the relation of the surface wind to the geostrophic wind is also dependent on these quantities. The curves in fig 5 show a maximum corresponding closely with the time of maximum

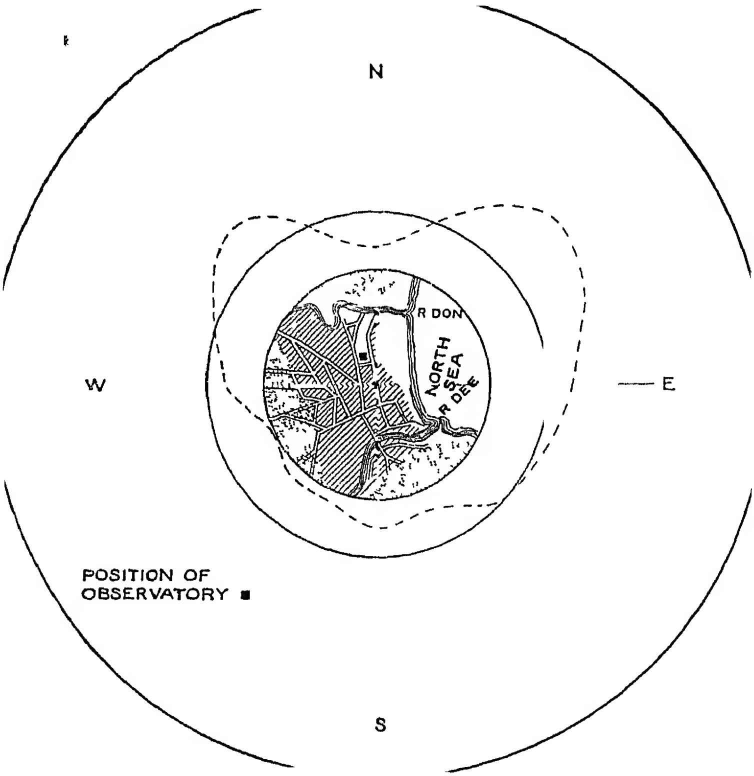


Fig. 6 —Relation between Geostrophic and Observed Surface Winds of Force 4 at Aberdeen

Central circle shows position of the city of Aberdeen with reference to the two river valleys and the sea Stippled area shows high ground Hatching area shows town buildings

The outer circle represents the gradient wind The inner circle represents 43 per cent of the gradient wind The dotted line represents the observed wind

temperature. Therefore if W represent the surface wind and G the geostrophic wind, the ratio W/G increases with increase of temperature, and vice versa. If then the surface layers be warmed or cooled from any cause whatsoever, we always find this effect on the ratio W/G.

The exposure of a station also has its effect on the ratio W/G.

TABLE I

The Relation of Surface Wind to Geostrophic Wind W/G expressed as a Percentage

Station	Nature of Site	Height of Ground	N	N N E	N E	E N E	E	E S E	S E	S S E.	S	S S W	S W	W S W.	W	W N W	N W	N.N W	Mean
Stornoway	Sloping shore of bay	51 ft	*48*	57	**63**	58	51	52	*48*	49	60	57	52	51	50	*48*	49	50	53
Malin Head	Headland knoll	208 ,,	72	**73**	60	39	65	66	51	*44*	46	48	48	56	57	67	70	68	58
Cahirciveen	Mouth of glen	30 ,,	58	57	55	47	55	42	39	38	*37*	40	41	54	54	**60**	59	57	50
Holyhead	Flat island	15 ,,	85	68	59	57	53	37	*34*	49	51	61	51	48	60	69	79	**87**	59
St. Mary's, Scilly	Hilly island	118 ,,	**74**	63	59	59	67	70	59	47	51	*45*	49	49	54	63	65	61	58
Portland Bill	Low point of headland	19 ,,	65	68	73	**80**	68	56	53	53	*48*	51	56	59	64	67	71	73	63
Aberdeen	College roof . ..	46 ,,	35	45	**61**	57	51	45	43	35	36	32	32	*31*	39	46	54	43	43
Spurn Head	Spit of sand	26 ,,	80	**85**	64	72	65	72	64	54	*49*	*49*	*49*	53	59	76	76	73	65
Paisley .	Inland station	—	37	33	53	**53**	47	33	*28*	35	33	35	36	36	41	36	42	47	38
Woburn .	Inland station	—	49	49	54	57	40	45	**61**	42	*31*	43	43	44	49	50	49	46	46

Note —The observations in every case refer to winds of force 4, and extend over 8 years. The highest and lowest percentages are indicated respectively by black type and italic figures.

Fig. 6 shows this effect on winds of force 4 at Aberdeen. The geostrophic wind is represented by the outer circle, while the dotted irregular curve gives the percentage which the surface wind is of the geostrophic wind. An idea of the exposure of the station is afforded by the circular portion of the ordnance map of the district placed at the centre of the figure. On the west side there is land, on the east, sea. To the south-west of the station lies the city, and we find that in this direction the surface wind has the lowest percentage, while in the north-easterly directions the percentages are largest. Towards the north-west lies the valley of the Don, and a fairly open exposure, the effects of which are also well brought out in the figure.

The effect of different exposures on winds of the same geostrophic magnitude will be understood readily from an examination of Table I.

It is evident, therefore, that no general rule can be given with regard to the value of the ratio W/G. It may have a wide range from approximately unity downwards, depending on the time of day, the season of the year, and the exposure of the station. In the same way the deviation α of the surface wind from the direction of the geostrophic wind is found to vary over a wide range.

An example of this is afforded by Table II, wherein are set out the values for Pyrton Hill and Southport, as given by J S Dines in the *Fourth Report on Wind Structure* to the Advisory Committee on Aeronautics.

It is necessary, therefore, in giving an estimate from the barometric gradient of the probable surface wind, as regards either direction or velocity, that due attention be paid to the details mentioned above

Occasionally the surface wind is found to be in excess of the gradient. This probably arises from a combination of a katabatic* effect with the effect of the pressure distribution, the katabatic effect more than compensating for the loss of momentum in the normal wind due to friction at the earth's surface.

Data for the purpose of examining the ratio W/G over the sea are very limited. The following table, as given in the Meteorological Office report for moderate or strong winds over the North Sea, will serve to show the deviation of the surface wind from the gradient wind, both in velocity and direction.

* I e Katabatic or Gravity Wind when the surface air over a slope cools at night or from any other cause it tends to flow down the slope, this is especially pronounced on clear nights In ravines, if snow-covered and devoid of forests, this wind often reaches gale force. Such a wind is known as a katabatic wind.

TABLE II

	Geostrophic Wind Direction	N	NNE	NE	ENE	E	ESE	SE	SSE	S	SSW	SW	WSW	W	WNW	NW.	NNW
		Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg	Deg
Pyrton Hill	Deviation α	29	24	11	13	22	27	31	37	40	35	37	43	37	42	45	34
Southport	,,	36	31	30	29	30	36	46	56	61	62	59	49	45	46	45	41

Mean surface-wind velocity (1) Pyrton Hill = 42 6 per cent of geostrophic wind.
(2) Southport = 51 6 ,, ,, ,,

Mean deviation of surface wind from geostrophic wind (1) Pyrton Hill = 32°.
(2) Southport = 44°.

TABLE III

RELATION OF SURFACE WIND TO GEOSTROPHIC WIND (BETWEEN 8·5 M/S AND 18 M/S) OVER THE NORTH SEA

1 Percentage Frequency of Points in the Veer of the Geostrophic Wind from the Surface Wind

Deviation in points ..	−4	−3	−2	−1	0	+1	+2	+3	+4	+5	+6	+7
N.W Quadrant .	0	1	0	2	21	32	**33**	14	1	0	0	0
S.W. ,, .	0	2	4	0	23	**30**	19	14	7	2	2	0
S E ,, ..	2	2	2	2	16	**29**	16	19	9	1	1	0
N E ,,	0	2	2	0	2	**31**	27	22	3	0	3	3

2. Percentage Frequency of the Ratios of Surface Wind to Geostrophic Wind within assigned limits

Limits of ratio .	0 24–0·36	0 36–0 48	0 48–0 60	0 60–0 72	0 72–0 84	0 84–0 96	0 96–01 08
N W ,, .	1	5	16	**38**	29	10	0
S W ,, ..	8	16	**34**	29	12	2	1
S E ,, ..	13	19	**26**	18	18	2	4
N E. ,, .	9	20	9	**29**	20	6	9

Note—The maximum values of both direction and velocity for each quadrant have been indicated by black figures.

Here again we see that no *definite* rule can be given for estimating
ıe surface wind from the geostrophic wind. It will be observed,
ɔwever, that each quadrant exhibits certain dominant features and
ust be considered therefore by itself. In this way a considerable
nount of guidance is obtained by the forecaster in estimating the
ind over the sea from a given pressure distribution.

We now pass to consider the actual wind in relation to the
ːostrophic wind in the first half-kilometre above the surface.
ʳ H. Dines, in his investigation on the relations between pressure
ıd temperature in the upper atmosphere, has found a high correla-
ɔn between the variations of these elements from their normal
lues for heights from 2 Km upwards. Below the 2-Km. level
e correlation coefficients gradually diminish until at the surface
actically no connection at all is found. Above the 2-Km. level
ɜ may regard the air as in an "undisturbed" condition, i e free
ɔm the effect of the friction at the earth's surface In this un-
sturbed region the velocity and direction of the wind at any given
ight are governed by the pressure and the temperature gradients
ling at that height, while in the lower layers we find considerable
viation from this law, evidently due to the effects of the surface
the earth on the air in contact with it

Several empirical formulæ have been given whereby the velocity
the wind at any height in the lower layers of the atmosphere may
· calculated from that at a definite height, say 10 m, above the
rth's surface From observations, up to 32 m, over meadow-
ıd at Nauen, Hellmann* confirmed an empirical formula $v = kh^{\frac{1}{5}}$,
ıich agrees very nearly with a formula $v = kh^{\frac{1}{4}}$ suggested by
·chibald† from kite observations in 1888. The results of observa-
ɔns up to 500 m., carried out in 1912 with two theodolites, are
ven by J. S. Dines in the *Fourth Report on Wind Structure* already
ferred to Here he has represented his conclusions by a series
curves, and in doing so has grouped the winds into three sets:
) very light, where the velocity at 500 m. is less than 4 m. per
cond, (2) light, with velocity between 4 m per second and 10 m.
r second; and (3) strong, with velocity greater than 10 m per
cond The curve for very light winds (see fig 7) shows that in
ıs class the surface wind approaches the geostrophic value, which
also marked for each group at the top of the diagram, much more
ɔsely than in any of the other groups. Curves of this type enable

* *Meteor Zeitschrift*, 1915.
† *Nature*, 27, p. 243.

one to judge of the average behaviour of the wind in the lowest half-kilometre according to the pressure gradient at the surface. When, however, curves are drawn for different hours of the day, 7 hr., 13 hr., 18 hr , these show differences among themselves even for the same surface gradient. A whole series of curves for various hours of the day and different seasons of the year would be necessary, therefore, before a complete solution of the problem could be obtained.

As these formulæ and curves just referred to are applicable under certain conditions only, and as the constants used differ for

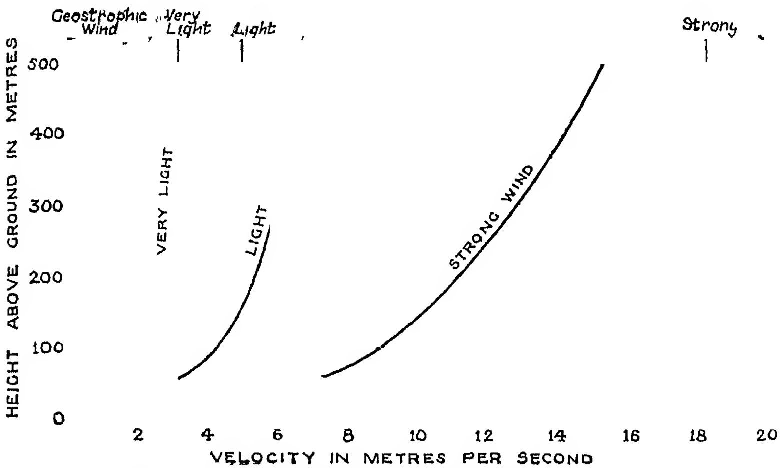


Fig 7—Change of Wind Velocity with Height within 500 metres above the surface

different times of the day and different seasons of the year, though they supply a rough working rule, yet a more exact solution of the problem is desirable. This has been supplied by the investigations of G. I. Taylor.* In his solution he regards the wind in the undisturbed layer as equivalent to the geostrophic wind at the surface, while the region between the surface and the undisturbed layer is considered as a slab through which the momentum of the undisturbed layer is propagated, as heat is conducted through a slab of material the two faces of which are kept at different temperatures. The momentum is propagated, according to the theory, by eddy motion, the surface of the earth acting as a boundary at which the momentum is absorbed The equation representing the propagation is given as

$$\rho \partial u/\partial t = \frac{\partial}{\partial z}\left(\kappa\rho\frac{\partial u}{\partial z}\right), \quad \ldots \ldots \quad (4)$$

* "Eddy Motion in the Atmosphere", *Phil. Trans.* A, **215**, p. 1 (1915)

ıere ρ = the densıty and κ = the "eddy conductivity" of the . For small heights up to 1 Km or thereby, ρ and κ are approxiıtely constant. Therefore the equation, representing the distrition of velocities with height and time within this region, may written as

$$\rho \partial u/\partial t = \kappa\rho \partial^2 u/\partial z^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The value of κ is, according to Taylor,* roughly $\frac{1}{2}wd$ where w the mean vertical component of the velocity due to the turbulence, d d represents approximately the diameter of a circular eddy.

The value of κ differs, however, according to (1) the nature of surface over which the air current is passing, (2) the season the year, and (3) the time of day. As both heat and momentum conducted by the eddies, the value of κ will be the same for th Values of κ have accordingly been determined by Taylor as lows.

(1) Over the sea (determined from temperature observations over the Banks of Newfoundland)	3×10^3 C.G S units.
(2) Over grassy land (determined from velocity observations by pilot balloons over Salisbury Plain)	5×10^4 C.G.S. units.
(3) Over land obstructed by buildings (determined from the daily range of temperature observations at different levels on the Eiffel Tower)	10×10^4 C G.S. units.

The effect of the season of the year on the value of κ is seen by mparing the values obtained from the Eiffel Tower observations January and June.

Whole range, 18 to 302 m.:

(1) January . ..	$4\,3 \times 10^4$	C G S. units
(2) June	$18\,3 \times 10^4$	,,
(3) Whole year	10×10^4	,,

That κ is also to a certain extent dependent upon the height ıy be understood by comparing the values for the first stage, to 123 m., with those for the last, 197 to 302 m

Mean value for the whole year.

(1) Lowest stage	15×10^4	C G.S units
(2) Highest stage	11×10^4	,,

The reason for this variation is to be found in the method used calculating κ. The nearer the ground the greater the daily

* *Proc Roy. Soc* A, 94, p 137 (1917)

variation of temperature, and therefore the error arising from the method used in the calculation is proportionately greater for the lower stages than for those higher up. The value of κ for the lowest stage is therefore not likely to be so accurate as that for the highest

From wind measurements Åkerblom* has deduced the value of κ for the whole range of the Eiffel Tower The value found, $7{\cdot}6 \times 10^4$ C.G.S. units, is in fairly good agreement with Taylor's mean value, and the agreement is sufficient to show that κ is the same both for heat and for momentum

This theory of eddy conductivity has been applied by Taylor in order to furnish an explanation of the diurnal variation of the velocity of the wind at the surface and in the lower layers of the atmosphere. Two important conclusions have been reached He has shown† that when once the steady state has been reached, a state which previous theories claiming to explain this diurnal variation took no account of, a relation can be found between the undisturbed wind (i e. the geostrophic wind), the surface wind, and the angle between the direction of the isobars and that of the surface wind. This relation takes the form

$$W/G = \cos\alpha - \sin\alpha, \qquad (6)$$

where W represents the surface wind, G the geostrophic wind, and α the angle between their directions The accuracy of this relation has been tested by comparing values of α observed by G. M. B. Dobson‡ with the calculated values for certain winds over Salisbury Plain. Some of these results are given in Table IV

TABLE IV.

$$W/G = \cos\alpha - \sin\alpha.$$

	Light Winds	Moderate Winds	Strong Winds
Observed value of W/G	0·72	0 65	0 61
α observed .	13 deg	$21\frac{1}{2}$ deg.	20 deg
α calculated . ..	14 ,,	18 ,,	20 ,,

* "Recherches sur les courants les plus bas de l'atmosphère au-dessus de Paris", *Upsala Soc. Scient Acta*, 2 (Ser 4), 1908, No 2.

† "Eddy Motion in the Atmosphere", *Phil Trans* A, **215** (1915). See note, p. 285.

‡ *Quar. Jour Roy Met. Soc.*, **40**, p 123 (1914).

The table shows that the agreement between observed and callated values is very close; with α greater than 45°, the equation, ›wever, no longer holds.

The other conclusion, as shown by Taylor,* on the assumption at the lag in the variation in wind velocity behind the variation turbulence which gives rise to it is small, is that the daily riation in turbulence is sufficient to explain qualitatively, and to certain extent quantitatively, the characteristics of the daily variaon in the wind velocity If the geostrophic wind G be reduced surface friction so that the direction of the surface wind is inclined an angle α to the "undisturbed" wind, then it is found that the ·ce of the surface friction, or the rate of loss of momentum to the rface, is given by $2\kappa\rho BG \sin\alpha$, where $B = \sqrt{\omega\sin\phi/\kappa}$. As before, is the angular velocity of the earth and ϕ the latitude The ation between this force of friction F and the velocity of the rface wind has also been examined by Taylor,† and found to be

$$F = 0{\cdot}0023\rho W^2.$$
$$\therefore\ 0\ 0023\ W^2 = 2\kappa BG \sin\alpha.$$

If now numerical values be given to ω and ϕ in $\kappa = \omega\sin\phi/B^2$ find that

$$\frac{1}{BG} = \frac{20{\cdot}4}{\sin\alpha}(\cos\alpha - \sin\alpha)^2, \qquad (7)$$

$\omega = 0\ 000073$, and $\sin\lambda = 0\ 77$, since for Salisbury Plain $= 50°$ N The values of $1/BG$ can therefore be found for a series values of α. Also from the same equation we see that κ/G^2 is unction of α If we tabulate the values of κ/G^2 for the same series values of α, we can find then the relations between α and κ These rious values are given in Table V (p. 276).

Basing his discussion upon these values of the constants, ıylor has constructed the curves given in fig 8. The abscissæ present the ratio of the wind velocity at any height to the geoophic wind, while the ordinates give the ratio of the height to geostrophic wind. If the geostrophic wind be 10 m per second, en the numbers for the ordinates will give the heights in dekametres, d those for the abscissæ the velocities in dekametres per second. ıe shape of each curve is determined by the value of α chosen, ch curve having its α value attached to it. Consequently where

* *Proc Roy Soc.* A, 94, p 137 (1917)
† *Proc. Roy. Soc.* A, 92, p. 198 (1916).

TABLE V

α, Degrees	$1/BG$, C.G.S. Units	κ/G^2, C.G.S. Units.
4	252	3·54
6	155	1·35
8	106	0·635
10	77·5	0·338
12	58·5	0·192
14	44·8	0·116
16	34·9	0·069
18	27·3	0·042
20	21·9	0·027
22	16·7	0·0156
24	12·9	0·0094
26	9·9	0·0055
28	7·4	0·0031
30	5·5	0·0017
32	3·7	0·00085
34	2·6	0·00038
36	1·7	0·00016

the geostrophic wind and the deviation at the surface are known, the curves enable us to determine the wind velocity at any desired height.

The curves may also be used to find the variation in velocity at a particular height under varying conditions of κ. The value of κ for the open sea, for Salisbury Plain, and for Paris we saw to be 3×10^3, 5×10^4, and 10×10^4 C.G.S. units respectively. If we take $\alpha = 10°$, then $\kappa/G^2 = 0·338$, which means that under these three conditions G must have the values 0·9, 3·8, and 5·4 m. per second respectively. Therefore the same geostrophic wind will suit different curves if the value of κ be altered, which it will be according to the exposure of the station, the season of the year, and the time of day.

From the foregoing we see how the wind at the surface and in the lowest layers differs considerably from the geostrophic value. As we ascend above the surface, a nearer approach is made to the geostrophic values, for the effect of surface friction diminishes with height. Turbulence also diminishes as we ascend, its influence being on an average very little felt at 1000 m., though on occasions it may reach to 2000 m.

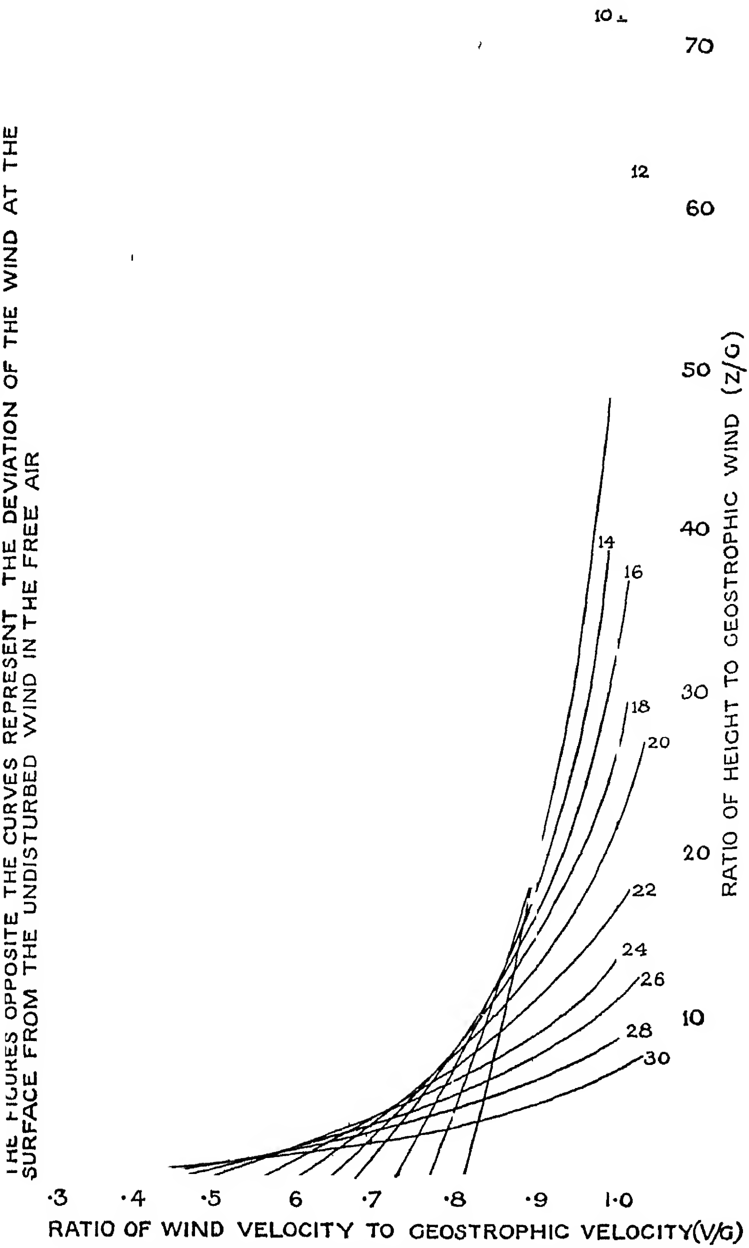


Fig 8 —Curves showing Variation of Wind Velocity with Height, according to the Theory of the Diffusion of Eddy-motion. (Taylor)

The spiral of turbulence affords another method of representing the variation with height of wind velocity in magnitude and direction In this method, first introduced by Hesselberg and Sverdrup * in 1915, when lines representing the wind velocity are drawn from the point at which the wind is measured, then their extremities lie on an equiangular spiral having its pole at the extremity of the line which represents the geostrophic wind. Thus in fig 8*a*, if O be taken as the origin and OgX the direction of the x-axis, Og represents the geostrophic wind G, OS the surface wind and $\angle$ SOg the angle α between the two. The wind at any height Z is represented by OP, and is the resultant of the geostrophic wind G and of another component represented by gP of magnitude $\sqrt{2}G \sin\alpha e^{-Bz}$ and acting in a direction which makes an angle $(\alpha + \frac{3\pi}{4} - Bz)$ with the

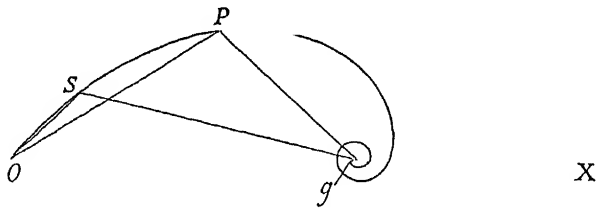


Fig 8*a*—Equiangular spiral representing velocity and direction of wind at any height

geostrophic wind. B has the same meaning as previously. An analysis of this method has been given by Brunt,† on the assumption that the coefficient κ is constant and that the geostrophic wind is the same at all levels. In a note added to the paper, Brunt also deals with the case where κ varies inversely as the height or inversely as the square of the height. The problem of κ varying as a linear function of the height has been considered by S. Takaya ‡ in a paper "On the coefficient of eddy-viscosity in the lower atmosphere". The solution enables the components of the wind to be calculated. The relations are equivalent to those found by Taylor (see Note 1). It must not be concluded, though the mathematical analysis appears to indicate it, that whenever a test is made on the wind that the results will produce an equiangular spiral. The gustiness of the wind prevents this, so that only when the mean of a large number of ascents is dealt with may one expect the wind values to form the equiangular spiral

* "Die Reibung in der Atmosphare" *Veroff d Geophys Inst d Univ Leipzig*, Heft **10**, 1915 † *Q J Roy. Meteor Soc*, **46**, 1920, p 175

‡ *Memoirs of the Imperial Marine Observatory, Kobe, Japan*, Vol IV, No. 1, 1930

The next region to be considered stretches from the surface to ıeight of approximately 8000 m.

Observations with pilot balloons indicate that the geostrophic ocity is reached on an average below 500 m., while the direction ıot attained until about 800 m. above the ground.* Each quadrant ɔws its own peculiarities, however. Thus Dobson† finds that for rth-east winds the gradient velocity is reached at 915 m., for ıth-east below 300 m, for south-west about 500 m., and for

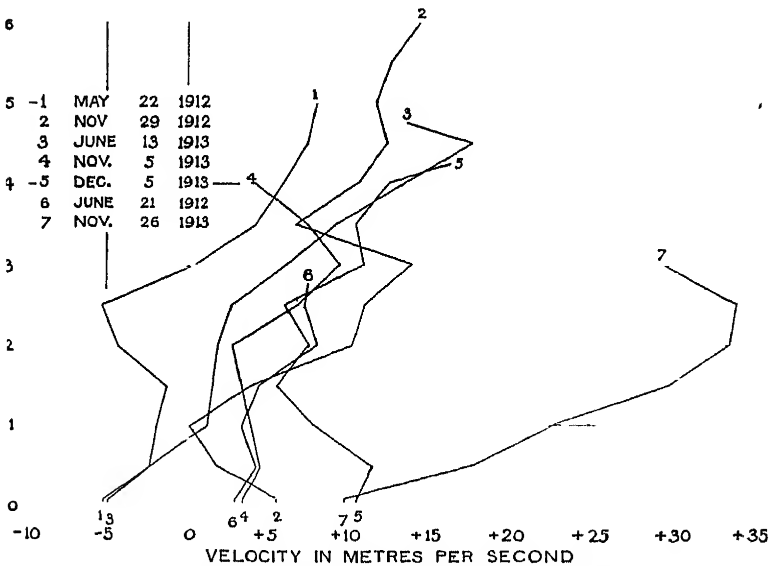


Fig 9—W to E Component of Wind Velocity on East Coast

th-west below 300 m Also the winds in the north-east and ıth-east quadrants show little or no increase after reaching the ıstrophic value, those in the north-east often showing a decrease, ıle those in south-west and north-west quadrants are marked by ontinual increase beyond the geostrophic value, the velocity in north-west being at 2500 m., 145 per cent of this value. The iation in direction also differs according to the quadrant. In north-east quadrant, even at 2500 m., Dobson finds that the ection is 6° short of the direction of the isobars On the other ıd, in the south-east quadrant the direction of the isobars is ched at 600 m, and above this level the wind veers still farther. e south-west winds behave somewhat similarly, only the gradient

* For theoretical treatment, see Note II, p 285a

† *Q. J. R. Met Soc*, 40, p 123 (1914).

direction is not attained until 800 m. is reached, while in the north-west quadrant the direction follows the isobars at 600 m. In this last quadrant, however, no further veer occurs until 1200 m. is reached, when a further veer begins. On the average the deviation of the surface wind from the gradient decreases from north-east to north-west, passing clockwise, Dobson's mean values being 27°, 24°, 19°, and 11° respectively.

These results refer to an inland station. When we come to

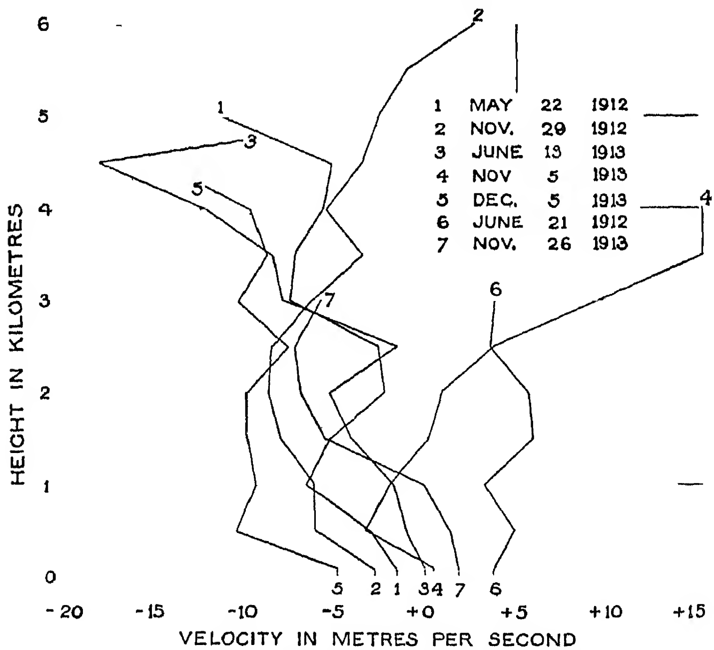


FIG. 10—S to N Component of Wind Velocity on East Coast

deal with a station on the coast we find even greater complications. Figs. 9 and 10, which represent the component velocities of a number of observations carried out with the aid of two theodolites at Aberdeen by the author,* serve to show the irregularities of these velocities. A greater variation is shown in the west-east component than in the south-north, as is to be expected from the exposure of the station. On the whole, the west-east component shows a tendency to increase with height, while any east-west velocity gradually dies out. The south-north diagram gives mainly negative values,

* *Q. J. R. Met. Soc.*, **41**, p. 123 (1915).

e. the winds observed had mainly a north-south component. This ɔmponent changes comparatively little in the first 4000 m., though ıgher up there is a tendency to increase indicated both for south-orth and north-south winds. This is in general agreement with ıe results arrived at by Dobson.

Cave, in his *Structure of the Atmosphere in Clear Weather*, has iven the results of observations carried out at Ditcham Park. Vhen we consider his results for heights between 2500 m. and 500 m., we find that there is a decided increase with height in ıe westerly components, the easterly components tending to die ut. On the other hand, both southerly and northerly components ıow an increase, the range at 7500 m being much greater than 2500 m., the actual values being from —20 m per second to -20 m. per second at the higher level, to —6 m. per second to -9 m. per second at the lower.

As the result of investigation, Cave has divided his soundings ı the troposphere * into five different groups, and has added a sixth ır winds in the stratosphere † These are:

(*a*) 1. "Solid" current; little change in velocity or direction.
2. No current up to great heights
(*b*) Considerable increase in velocity.
(*c*) Decrease of velocity in the upper layers.
(*d*) Reversals or great changes in direction
(*e*) Upper wind blowing outward from centres of low pressure: frequently reversals at a lower layer.
(*f*) Winds in the stratosphere.

In group (*a*) the gradient direction and velocity are reached ırly, and thereafter the wind remains nearly constant. There is ractically no temperature gradient, and the pressure distribution t different heights is similar to that at the surface.

Group (*b*) is mainly due to a westerly or south-westerly type, ıd represents the average conditions where depressions are passing ıstwards over the British Isles. There is here a marked tempera-ıre gradient over the area.

In group (*c*) are included mainly easterly winds, the pressure

* Troposphere, i e the part of the atmosphere in which the temperature falls off ith increasing altitude In latitude 55° it extends from the surface up to about Km , in the tropics it extends to about 17 Km.

† Stratosphere, i.e the external layer of the atmosphere in which there is no ɔnvection It lies on the top of the troposphere, and the height of its base above ıe surface varies from equator to poles (see troposphere). The temperature ıanges within it are in a horizontal direction.

distribution showing an anticyclone to the north. The gradient velocity is reached at about 500 m., the gradient direction at a point a little higher. Thereafter decrease in velocity takes place and occasionally a *backing* of the wind, though the latter does not invariably occur.

With " reversals ", which are placed in group (*d*), the surface wind is almost always easterly; the upper, westerly or south-westerly

Here we have a warm current passing over a colder, and the result is generally rain. Very often in summer there is found a south-west current passing over a south-east, the two being associated with shallow thunderstorm depressions, the south-west current supplying the moisture to form the cumulo-nimbus clouds.

From an examination of the winds in group (*e*), it is almost always found that the depressions from which the winds come advance in the direction of the upper air current. This is particularly the case with north-westerly upper winds. With south-westerly upper winds we have very often conditions similar to those mentioned under (*d*), with corresponding results.

Observations within the stratosphere are comparatively few, but, in general, they show that the wind within this region tends to fall off with increase in height, and that the direction is almost invariably from some point on the west side of the north-south line.

Several models, which show at a glance how the air currents change with height, have been constructed by Cave For a description of these and a full account of his investigation the reader is referred to his book already mentioned.

Let us now examine the wind structure in these upper regions of the atmosphere from the theoretical standpoint.

We have already noted that the variations in the distribution of pressure in the upper atmosphere are closely correlated with the variations in the temperature distribution. Starting with the ordinary equation for the diminution of pressure with height and combining it with the characteristic equation for a permanent gas, we are able to find an equation giving the variation of pressure gradient with height. These equations are:

$$\frac{\partial p}{\partial z} = -g\rho, \qquad (8)$$

and

$$p/T = R\rho,$$

the latter giving

$$\frac{\partial \rho}{\rho} = \frac{\partial p}{p} - \frac{\partial T}{T}. \qquad (9)$$

lso if the horizontal pressure and temperature gradients be written
s $\frac{\partial p}{\partial x} = s$, and $\frac{\partial T}{\partial x} = q$, respectively, then we have

$$\frac{\partial s}{\partial z} = \frac{\partial^2 p}{\partial x \partial z},$$

e. from equation (8)

$$\frac{\partial s}{\partial z} = -g\frac{\partial \rho}{\partial x}. \quad\quad (10)$$

'herefore combining (9) and (10) we have for the change with eight in pressure gradient,

$$\frac{\partial s}{\partial z} = -g\rho\left\{\frac{1}{p}\frac{\partial p}{\partial x} - \frac{1}{T}\frac{\partial T}{\partial x}\right\}$$

$$= g\rho\left\{\frac{q}{T} - \frac{s}{p}\right\} \quad\quad (11)$$

To find numerical values we must substitute for ρ its value /RT. For dry air R = $2{\cdot}869 \times 10^6$ C.G S. units, while for air ıturated with water vapour at 273*a* its value is, $2\ 876 \times 10^6$ C.G S. nits This differs only slightly from the value for dry air Also ıe uncertainties which arise in connection with the determination f the wind velocity in the upper air are greater than the variations ı R, and therefore the value for dry air may be used on all occasions 'ithout any appreciable error With this value, and with g as 81 cm /sec.^{2}, we have

$$\frac{\partial s}{\partial z} = 3{\cdot}42 \times 10^{-4}\frac{p}{T}\left\{\frac{q}{T} - \frac{s}{p}\right\}. \quad\quad (11a)$$

f now we express the variation in pressure in millibars per metre f height and take the gradients in pressure and temperature over oo Km., the rate of increase of pressure gradient per metre of eight in millibars per 100 Km. is

$$3{\cdot}42 \times 10^{-2}\frac{P}{T}\left\{\frac{Q}{T} - \frac{S}{P}\right\},$$

and S being expressed in millibars, T and Q in degrees absolute

The variation in pressure gradient depends therefore on the ıfference of the quantities Q/T and S/P. Now T falls from about 80*a* at the surface to approximately 220*a* at 9 Km., whereas P hanges from 1010 millibars to nearly 300 millibars within the

same range We see, therefore, that S/P runs through a considerable range of values, while Q/T remains comparatively constant. The variation in pressure difference is therefore not constant within the region considered, but is likely to show positive values at first, then change through zero to negative values higher up. If the pressure difference remained constant up to 9 Km., then $V\rho$ would be constant, and the velocity of the wind would increase in inverse proportion to the density of the air as we ascend. Now Egnell believed that he found by observation of clouds that $V\rho$ actually was constant, and in consequence this law, $V\rho$ = a constant, has been termed Egnell's Law. We have seen, however, that the observations by pilot balloons do not confirm the law. The wind very often shows an increase in velocity with increase in height, especially winds with a westerly component, but this increase is generally less, even in the latter case, than in accordance with a uniform gradient. Equation (11) is therefore much more in agreement with the behaviour of the actual winds than a constant pressure gradient would be.

The variation of wind with height can now be obtained by combining equation (11) with the relation

$$s = 2v\rho\omega \sin\phi, \text{ or, } v\rho = s/2\omega \sin\phi \quad \ldots\ldots (12)$$

Let v be the component of the wind velocity parallel to the y-axis drawn towards the north, the x-axis being drawn towards the east. Then

$$\rho\frac{\partial v}{\partial z} + v\frac{\partial \rho}{\partial z} = \frac{\partial s}{\partial z}/2\omega \sin\phi,$$

$$\text{i.e. } \frac{1}{v}\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{1}{s}\frac{\partial s}{\partial z},$$

$$\text{or} \quad \frac{1}{v}\frac{\partial v}{\partial z} = \frac{1}{s}\frac{\partial s}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{g}{s}\frac{\partial \rho}{\partial x} - \frac{1}{\rho}\frac{\partial \rho}{\partial z}$$

$$= -\frac{g\rho}{s}\left(\frac{1}{p}\frac{\partial p}{\partial x} - \frac{1}{T}\frac{\partial T}{\partial x}\right) - \left(\frac{1}{p}\frac{\partial p}{\partial z} - \frac{1}{T}\frac{\partial T}{\partial z}\right)$$

$$= -\frac{g\rho}{s}\left(\frac{s}{p} - \frac{q}{T}\right) + \frac{g\rho}{p} + \frac{1}{T}\frac{\partial T}{\partial z}$$

$$= \frac{1}{T}\left\{\frac{g\rho q}{s} + \frac{\partial T}{\partial z}\right\},$$

$$\text{i.e. } \frac{s}{v}\frac{\partial v}{\partial z} = \frac{1}{T}\left(g\rho q + s\frac{\partial T}{\partial z}\right);$$

and as $\quad s/v = \rho 2\omega \sin\phi,$

$$\therefore \frac{\partial v}{\partial z} = \frac{1}{\rho T 2\omega \sin\phi}\left\{-q\frac{\partial p}{\partial z} + s\frac{\partial T}{dz}\right\}$$

$$= \frac{1}{2\omega \sin\phi \rho T}\left\{\frac{\partial p}{\partial x}\frac{\partial T}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial T}{\partial x}\right\} \quad \dots \quad (13)$$

ɔng the x-axis the corresponding value will be

$$\frac{\partial u}{\partial z} = -\frac{1}{2\omega \sin\phi \rho T}\left\{\frac{\partial p}{dy}\frac{\partial T}{dz} - \frac{\partial p}{\partial z}\frac{\partial T}{\partial y}\right\}. \qquad (13a)$$

re the negative sign must be used because if the pressure rease towards the north then the wind will be from the *east*. If then the wind be observed at various levels it is possible from se equations to calculate the separation of the isobars and iso-rms at the different levels. For this purpose it is necessary to ɔw the values of p and T for each level considered. In any parti-ar case the normal values of these quantities for the month in ich the observation takes place may be taken without any very ious error. We may then proceed to calculate the separation of isobars and isotherms at intervals of a kilometre in the following y

The change of pressure difference we already expressed in the m

$$\frac{\partial s}{\partial z} = 3\ 42 \times 10^{-4}\frac{p}{T}\left\{\frac{q}{T} - \frac{s}{p}\right\} \text{ in C G S units}$$

hen the pressure be expressed in millibars and the temperature degrees absolute, the change of pressure-difference per kilometre height may be written as

$$\Delta s = 34{\cdot}2\frac{P}{T}\left(\frac{\Delta T}{T} - \frac{\Delta P}{P}\right), \dots\dots\dots\dots \quad (14)$$

ere ΔP and ΔT are the horizontal changes per 100 Km in ssure and temperature respectively. The wind velocity W due this pressure difference ΔP can be found from the relation

$$W = \frac{R}{2\omega\sin\phi}\frac{T}{P}\Delta P = K\frac{T}{P}\Delta P \quad \dots \qquad (15)$$

U and V be the components of W from west to east and n south to north respectively, then the components of pressure

difference at any level as deduced from the wind observations a

$$\left.\begin{aligned} \Delta_N P &= \frac{1}{K}\frac{P}{T}U \\ \text{and} \qquad \Delta_W P &= \frac{1}{K}\frac{P}{T}V \end{aligned}\right\}\ldots\ldots\ldots\ \ldots\ldots(15a)$$

Similarly the components of temperature difference can b expressed from equation (14) in the form

$$\left.\begin{aligned} \Delta_N &= T\frac{T}{P}\left(\frac{\Delta s}{34\cdot 2}\times T + \Delta_N P\right) \\ \text{and} \qquad \Delta_W &= T\frac{T}{P}\left(\frac{\Delta s}{34\cdot 2}\times T + \Delta_W P\right) \end{aligned}\right\}\ldots\ldots\ldots(14a)$$

Table VI (p. 287) is an example of the application of thes equations. Of the last four columns the first two give the separa tion in kilometres between the component isobars where th difference is 1 millibar, the second two between the componen isotherms where the difference is 1° C When the direction of th resultant isobars and isotherms for the various levels are calculate we find the following directions

Height in Km	0	1	2	3	4
Isobar from	270°	272°	271°	214°	263°
Isotherm from	—	15°	271°	210°	334°

This appears to indicate the approach of a warmer current fron the south-west at a height over 3 Km. The pressure distributio at 7 hr. on the 28th gave a depression over Iceland with a surfac temperature of 50° F. The increase of temperature indicated a the 3000-m. level is apparently due therefore to the warm air fron this depression pushing its way across the colder northerly current This seems to be in agreement with Bjerknes' theory* of the circu lation of air within a cyclone.

The observations we have been considering hitherto refer to one station only, so that we have obtained only a very small section of the isobars and isotherms for the different levels. If a number of obser- vations be made simultaneously at different stations over the British

* *Q. J. Roy. Met. Soc*, 46, p. 119.

TABLE VI

COMPUTATION OF PRESSURE AND TEMPERATURE DISTRIBUTION FROM PILOT BALLOON ASCENT OF 28TH MAY, 1914

Height in Km	Vel, m p s	Direction	W-E Component	S-N Component	$\Delta_N P$	$\Delta_W P$	$\Delta_N T$	$\Delta_W T$	$\frac{100}{\Delta_N P}$	$\frac{100}{\Delta_W P}$	$\frac{100}{\Delta_N T}$	$\frac{100}{\Delta_W T}$
0	0·0	·	− 0·0	+ 0·0	+ 0·60	0·0			+ 167	..	..	..
1	1·7	291°	+ 1·6	− 0·6	+ 0·22	− 0·8	− 0·89	+ 0·23	+ 455	− 1250	− 112	− 435
2	2·9	288°	+ 2·8	− 0·9	+ 0·35	− 0·10	+ 0·48	− 0·09	+ 286	− 1000	+ 208	− 1111
3	7·0	231°	+ 5·3	+ 4·4	+ 0·59	+ 0·49	+ 0·95	+ 1·96	+ 169	+ 204	+ 103	+ 51
4	8·9	263°	+ 8·7	+ 1·1	+ 0·87	+ 0·11	+ 1·32	− 1·18	+ 115	+ 909	+ 76	− 85

Isles, say, then a series of maps may be drawn showing the air flo
at each level. These will afford an indication of the distribution c
pressure and temperature at the various levels. In fig. 11 the pressur
distribution at the surface at 18 hr. on 7th September, 1922, is give
in (*a*). The following four members of the series show approximatel
the run of the isobars at the levels indicated as deduced from pilc
observations made at 17 hr., while the last of the series depicts th
pressure distribution at the surface at 7 hr. on the following morning
The separation of the isobars is 2 millibars in every case. At th
surface the isobars run from north-east to south-west, but higher u
the direction changes towards a north to south direction. Thi
appears to indicate a mass of rather warmer air towards the wes
or south-west, especially about the 6000-ft. level. The velocities
however, are comparatively small and therefore a break-up of th
system is not to be expected. Instead, as the 7 hr. chart of th
following morning shows, there has taken place a further development
and the direction of the isobars at the surface has now become mucl
more in accordance with the upper air isobars of the previous evening

We must now consider the case of curved isobars. In th
expression for the gradient wind determined near the beginning o
our survey there were found to be two parts, one dependent upor
the rotation of the earth, the other on the curvature of the path
Hitherto we have dealt only with the first part, but now we shal
consider briefly the effect of the curvature of the path upon the
relation of the wind to the distribution of pressure. In discussing
the circulation of air in temperate latitudes, Shaw arrives at the
following conclusion * "Thus out of the kaleidoscopic features of
the circulation of air in temperate latitudes two definite states sort
themselves, each having its own stability. The first represents air
moving like a portion of a belt round an axis through the earth's
centre. It is dependent upon the earth's spin, and the geostrophic
component of the gradient is the important feature; the curvature of
the isobars is of small importance. The second represents air rotating round a point not very far away: it is dependent upon the local
spin, and the curvature of the isobars with the corresponding cyclostrophic component of the gradient is the dominant consideration."

Up to the present we have been considering only one point in the path of the air, and the lines of flow of the air at that point we have regarded as coincident with the isobars in the upper air and making a definite angle with them near the surface owing to the

* *Manual of Meteorology*, Part IV, p. 236.

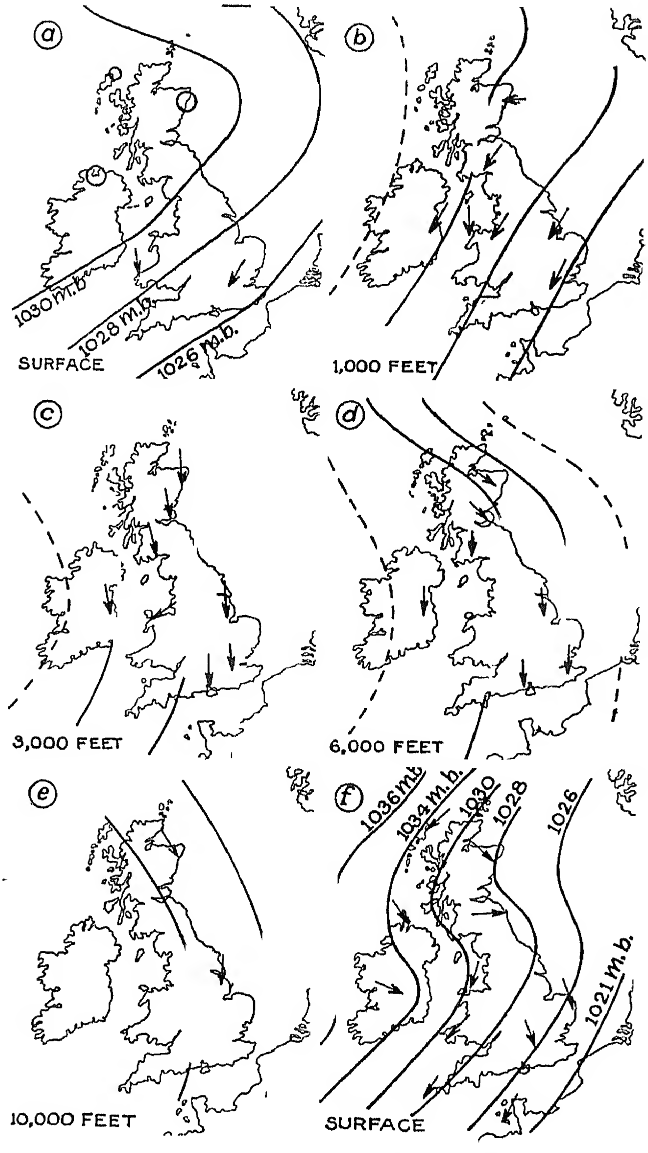


FIG 11.—Map showing the Pressure Distribution and Wind Direction at the Surface at 18 hr on 7th September (*a*), at 7 hr on 8th September (*f*), and (*b* to *e*) the approximate Direction of the Isobars at 1000 ft, 3000 ft, 6000 ft, and 10,000 ft, as deduced from Pilot Balloon Observations at 17 hr. on 7th September, 1922.

turbulence in the atmosphere. When we come to consider a succession of states, however, we see that the paths of the air are not necessarily coincident with the lines of flow or the latter with the isobars. In the case of the first state mentioned above by Shaw the isobars are straight and the paths of the air are coincident with the lines of flow; but when the two states are superposed and a series of maps drawn giving the pressure distribution at definite intervals it is seen that the paths of the air are no longer coincident with the lines of flow or with the isobars. What then are the paths of air in a cyclone?

A partial solution of the problem may be reached after the following manner. It is a well-known fact of experience that one of the characteristics of a cyclone is that it travels across the map, and when the isobars are circular that the velocity of translation is rapid. We shall here confine ourselves therefore to the examination of a circular, rapidly moving storm, termed the "normal" cyclone or cartwheel depression.

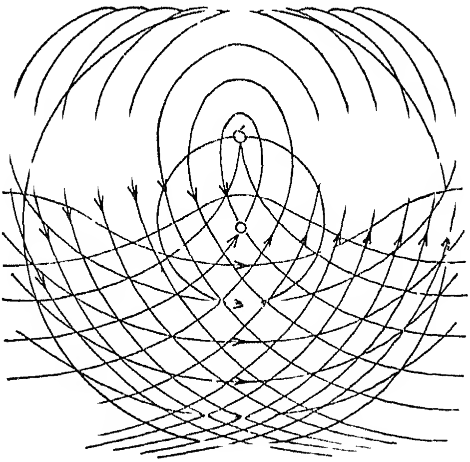


Fig 12.—Trajectories of Air for a Normal Cyclone (from Shaw's *Manual of Meteorology*)

If two horizontal plane sections of a normal cyclone be taken, including between them a thin lamina or disc of rotating air, and if this disc travel unchanged in a horizontal direction, then to obtain the actual velocities of any point on the disc the velocity of translation must be combined with the velocity of rotation. If the velocity of translation be V and the angular velocity n, then the centre of instantaneous rotation will be distant from the centre of the disc a distance V/n. This centre of instantaneous rotation will travel in a line parallel to the line of motion of the centre of the disc. The actual paths of the air particles are traced out by points attached to a circle which rolls along the line of instantaneous centres and whose radius $= V/n$. Fig. 12 represents these trajectories. The figure shows, in one position, the circle of radius V/n, its centre O, and the instantaneous centre O′. The circle rolls on

line through O′ perpendicular to OO′. The path of a particle a cusp, a loop, or neither, according as the tracing point is on, iout, or within the circle.

From this we see that in the normal cyclone there are two centres, one, O (fig. 12), the actual centre of the rotating disc which is ied the *tornado* centre, the other, O′, the centre of instantaneous tion termed the *kinematic* centre. They lie on a line perpendi-r to the path of the cyclone, and are distant from each other by length V/n.

In such a system as this the isobars will not coincide exactly ı the lines of flow. For the present neglecting the variation in ude and in density, and also the curvature of the earth's surface, shall regard the cyclone as moving along a horizontal plane. hat case the system of isobars will be obtained by compounding stem of circular isobars embedded in a field of straight isobars. centre of the circular system will not coincide with either of centres already referred to, but will be at a distance from the matic centre $= V/(2\omega \sin\phi + n)$ and lie on the line joining the matic and tornado centres. This centre has been termed the *mic* centre, and is the centre of the isobaric system as drawn map It is therefore quite easily identified, but one must bear ind that it is not the *only* centre in a normal cyclone

f we take the centre of the rotating disc as origin, with x and y towards the east and north respectively, then for an eastward ›ity of translation V, the pressure will diminish uniformly towards north at the rate $2\rho V\omega \sin\phi$, i.e. the field of pressure will be ›sented by

$$\int_{p'_0}^{p'} dp = -\int_0^y 2\rho V\omega \sin\phi dy,$$

e p' = pressure at any point, and p'_0 = pressure at any point ie x-axis;

$$\text{i.e. } p' - p'_0 = -2\rho V\omega \sin\phi y \ldots \qquad \ldots(16)$$

'or the circular field with its centre at the origin we have

$$\int_{p_0}^{p} dp = \int_0^r (2\rho v\omega \sin\phi + v^2\rho \cot\alpha/R) dr,$$

e p is the pressure at any point distant r from the origin, and e pressure at the origin.

If we neglect the curvature of the earth, then $v^2 \cot\alpha/R = v^2/$
Also $v = rn$.

$$\therefore \int_{p_0}^{p} dp = \rho n \int_0^r (2\omega \sin\phi + n)\, r dr,$$

$$\text{i.e. } p - p_0 = \rho\frac{n}{2}(2\omega \sin\phi + n)\, r^2 = \rho\frac{n}{2}(2\omega \sin\phi + n)(x^2 + y^2) \quad (1$$

By combining the two equations (16) and (17), we have for tl resultant field

$$P - P_0 = \frac{\rho n}{2}(2\omega \sin\phi + n)(x^2 + y^2) - 2\rho V\omega \sin\phi y \ldots (18)$$

This represents a circular field of pressure round a point

$$x = 0, \; y = \frac{2\omega \sin\phi V}{n(2\omega \sin\phi + n)},$$

and P_0 is the pressure at the centre of this field and not at the origi

Now the distance of the kinematic centre from the tornac centre which was chosen as origin is equal to V/n Therefore tl distance of the kinematic centre from the dynamic centre is

$$V/n - \frac{2\omega \sin\phi}{2\omega \sin\phi + n} \times \frac{V}{n} = V/(2\omega \sin\phi + n).$$

We see, therefore, that this combination of a field of straight isobai with a circular system embedded in it is sufficient to give the fiel of pressure necessary to keep the disc rotating.

In the normal cyclone it follows that the centre of low pressui being not the centre of the lines of flow, the wind possesses a defini counter-clockwise velocity at the centre of low pressure. When a actual example of a rapidly moving circular storm such as that 10th–11th September, 1903, is examined, we find that the syste actually does possess such a wind, and consequently this diverge wind, which has often been regarded as accidental, is in reality perfect agreement with the pressure system. Another feature whi the normal cyclone possesses in common with an actual circul cyclone is the greater incurvature in the rear of the cyclone as cor pared with that in front.

" From these considerations," says Shaw,* " we are led to acce the conclusion to be drawn from the conditions of the normal cyclon

* *Manual of Meteorology*, Part IV, p. 245

mely, that the wind *calculated from the gradient by the full formula 'ng the curvature of the isobars, gives the true wind in the free air t at the point at which the gradient is taken but at a point distant om it along a line at right angles to the path and on the left of it by e amount $V/(2w \sin\phi + n)$.*"

The calculated trajectories in the case of the normal cyclone

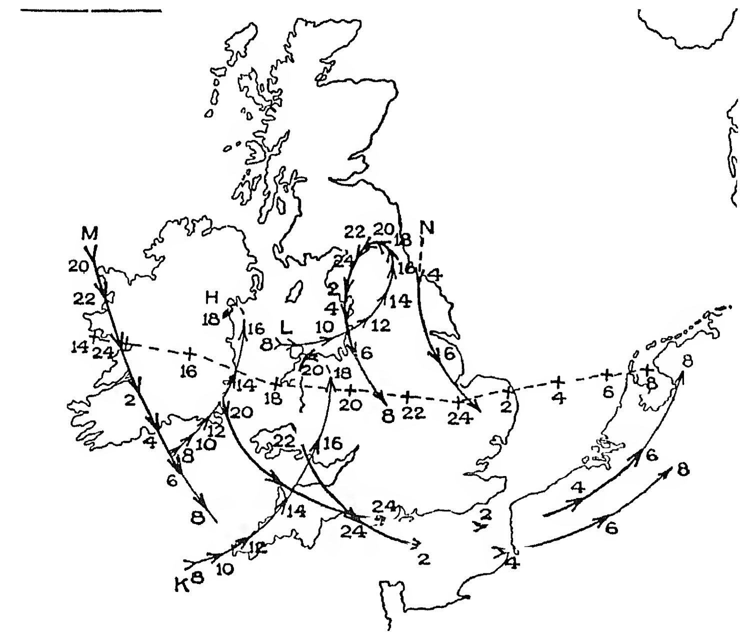


Fig 13.—Trajectories of Air in Circular Storm, 10th to 11th September, 1903

ve already been referred to and given in fig. 12. A comparison these with the actual trajectories for the storm of 10th–11th ptember, 1903 (fig. 13), shows at once the remarkable similarity tween the two sets, indicating still further that this actual cyclone d the normal cyclone are very close akin the one to the other. e trajectories of the September cyclone are reproduced from *e Life History of Surface Air Currents.**

Here we have considered in a very fragmentary way only one m of stable rotation, namely the circular. Even then no account s taken of the discontinuity of velocity which must occur at the

* *Life History of Surface Air Currents*, by W. N. Shaw and R. G. K. Lempfert, O., No. 174, London, 1906.

edge of the rotating disc in the normal cyclone, but it is possible show that this discontinuity can be accommodated by includi in the revolving column of air an outer region represented by t law of the simple vortex with vr constant. The regions beyor what have hitherto been included in the revolving disc will al form part of the cyclone, therefore, and not simply belong to t environment.

For further treatment of this subject the reader is referred treatises on dynamical meteorology, such as Shaw's *Manual Meteorology*, as in this brief study some of the intricacies only of t problems of wind structure, rather than their solutions, have bee placed before him.

Note I

The equations of motion of air over the surface of the eart when the effect of eddy viscosity is taken into account are, whe the steady state has been reached and the motion is horizonta

$$0 = -2\omega v \sin\phi + \kappa\frac{d^2u}{dz^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$0 = +2\omega u \sin\phi - 2\omega G \sin\phi + \kappa\frac{d^2v}{dz^2} \quad \ldots\ldots (2)$$

On eliminating u from the above we find that the equation fo v becomes

$$\frac{d^4v}{dz^4} + \frac{4\omega^2 \sin^2\phi}{\kappa^2}v = 0,$$

or
$$\frac{d^4v}{dz^4} + 4B^4v = 0, \text{ where } B^2 = \frac{\omega \sin\phi}{\kappa}.$$

Now v does not become infinite for infinite values of z, and th solution of the equation is therefore

$$v = A_2e^{-Bz}\sin Bz + A_4e^{-Bz}\cos Bz \quad \ldots\ldots\ldots (3)$$

On differentiating this value of v twice with respect to z, an substituting in (2), we get

$$u = G + A_2e^{-Bz}\cos Bz - A_4e^{-Bz}\sin Bz \quad \ldots\ldots\ldots (4)$$

Now G = value of the gradient wind velocity, and therefore for great heights $u = G$, i.e. the gradient wind velocity and $v = 0$.

The values of A_2, A_4 are found by imposing suitable boundary iditions.

Now at $z = 0$ these are

$$\left[\frac{du/dz}{u}\right]_{z=0} = \left[\frac{dv/dz}{v}\right]_{z=0}, \text{ and } \tan\alpha = -\left[\frac{v}{u}\right],$$

ere α = angle between the observed wind and the gradient id.

From these conditions

$$A_2 = \frac{-\tan\alpha(1 + \tan\alpha)}{\tan^2\alpha + 1}G; \; A_4 = \frac{-\tan\alpha(1 - \tan\alpha)}{\tan^2\alpha + 1}G.$$

The surface wind $W = \sqrt{[u^2 + v^2]_{z=0}}$

$$= \sqrt{A_4^2 + (A_2 + G)^2}$$

$$= \frac{G}{1 + \tan^2\alpha}\sqrt{\tan^2\alpha(1 - \tan\alpha)^2 + (1 - \tan\alpha)^2}$$

$$= G\frac{(1 - \tan\alpha)}{\sec\alpha} = G(\cos\alpha - \sin\alpha),$$

ɔr $\quad W/G = (\cos\alpha - \sin\alpha).$

Note II

The theory of eddy motion also accounts for the observed fact t the magnitude of the gradient wind is reached at a level lower n that at which the gradient direction is attained

The height at which the gradient direction is reached is found m equation (3) of Note I by putting $v = 0$ If H_1 be the ght, then

$$0 = A_2 \sin BH_1 + A_4 \cos BH_1,$$

l therefore $\quad \tan BH_1 = -\dfrac{A_4}{A_2}.$

Substitute the values of A_4 and A_2 already found, and we obtain

$$\tan BH_1 = -\frac{1 - \tan\alpha}{1 + \tan\alpha} = \tan\left(\alpha - \frac{\pi}{4}\right).$$

Since α is positive and less than $\frac{\pi}{4}$, the smallest value of H_1 is got from

$$BH_1 = \frac{3\pi}{4} + \alpha. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The height H_2, at which the value of gradient velocity is reached, is given by $u^2 + v^2 = G^2$, which, on substitution, becomes

$$e^{-BH_2} = \frac{(1 + \tan\alpha)\cos BH_2 - (1 - \tan\alpha)\sin BH_2}{\tan\alpha} \quad \ldots(2)$$

From this equation BH_2 can be found in terms of $\tan\alpha$

The following table, given by Taylor, shows the values of BH_1, BH_2, and H_1/H_2 as α goes from 0° to 45°.

α.	BH_1	BH_2	H_1/H_2
0	2 35	0 78	3 0
20	2·70	1·04	2 6
45	3 15	1 44	2·2

Now for Salisbury Plain Dobson found that the deviation α was, in a large number of cases, 20°; also that $\frac{H_1}{H_2}$ for this deviation was $\frac{800 \text{ metres}}{300 \text{ metres}} = 2\ 66.$

This is in good agreement with the theoretical value 2·6.

CHAPTER IX

ubmarine Signalling and the Transmission of Sound through Water

Although practically every other branch of science has had siderable technical application, that of acoustics has until the few years remained practically in the academic stage, and few n among scientific men gave serious attention to it Bells, gs, whistles, sirens, and musical instruments have indeed been d from remote times both for enjoyment and for signalling pur-es, but their development has mainly been on empirical lines, h but little assistance from the physicist.

The Great War has, however, brought about a striking change this as in many other directions, and acoustics is now becoming only an important branch of technology, but shows signs even developing into the engineering stage and giving us a new and verful method of power transmission, to judge by the pioneer k of M. Constantinesco, who has already developed it for the ration of rock drills and riveting machines, and shown how it y be applied to motors and other machines. Few branches of ence now offer such possibilities to the inventor.

Acoustic signalling is of especial importance in connection with igation, as sound is the only form of energy which can be trans-ted through water without great loss by absorption. The rela-ly high electrical conductivity of water renders it almost opaque light and to electromagnetic waves.

The present article deals principally with acoustic signalling ler water, but certain allied problems, such as sound ranging, th sounding, and other applications to navigation, will also be efly referred to.

As is well known, sound consists of a vibratory disturbance of aterial medium, such as a gas, solid, or liquid, and its phenomena

e almost of a purely mechanical nature. When a bell or tuning ›rk is struck it is thrown into vibration, and as any part moves ›rward it compresses the medium in front of it and also gives it forward velocity. As the vibration reverses so that the movement is in the opposite direction, the mass or inertia of the medium ›eeps it moving forward, and a partial vacuum or rarefaction is ·oduced, until the vibration again reverses and forms a fresh ›mpression. These regions of compression and rarefaction there-re travel forward as a series of pulses or waves away from the urce in much the same way as ripples are formed on the surface a pond when a stone is dropped into it. In the case of the rface ripples, however, the real motion of the water is partly up d down, or transverse to the direction of movement of the waves, ›ereas in the case of sound the motion of each particle of the edium is mainly forwards and backwards along the line of propa-tion of the sound. Sound vibrations are therefore spoken of as ngitudinal or in the direction of transmission, which differentiates em from all other kinds of vibrations, such as those of ordinary ives, light, or electromagnetic waves, which are said to be trans-rse. It at once follows from this that although many of the entific principles of optics can be and indeed have been success-ly applied to sound, there can be nothing in acoustics correspond-; to polarization in light. This point is made clear at the outset, the phenomena of light are fairly generally known and are most mulating to acoustic development.

Fundamental Scientific Principles

In order to understand the operation of modern acoustic trans-tting and receiving instruments properly, it will be well to start h a brief statement of certain scientific principles and definitions. ne of these are well known, but others require a few words of ›lanation.

Sounds are divided into musical notes and noises, and musical es are spoken of as differentiated by intensity, pitch, and *timbre* quality. A musical note is produced whenever the vibrations are i regular character, so that each wave is similar to the previous . The intensity or loudness of the note depends on the strength amplitude of the vibration, its pitch on the number of vibrations second or frequency, and its *timbre* or quality on the form the vibration. The purest musical note is given by uniform

vibrations of a simple harmonic character, and if the wave-form is saw-toothed or shows any other variation from the sine form, the note is more or less piercing in quality, due to the presence of overtones or higher harmonics besides the fundamental pure tone.

Noises are differentiated from musical tones by having no regular character, and are made up of a number of vibrations of different intensity and pitch. Speech may be described as a noise from the point of view of acoustic transmission and reception, on account of the variable nature of the vibrations; and also the sound from machinery, ships, &c. This is a serious difficulty as regards the detection and recognition of such sounds, as nearly all transmitting and receiving devices are more or less "selective" in character, i.e. they respond better to certain definite frequencies and are relatively insensitive to others. Everyone knows that telephones or gramophones reproduce certain sounds better than others, and acoustic signalling, like wireless transmission, is far more effective with "tuned" devices, which are, however, very insensitive to other frequencies.

Velocity of Propagation

An accurate knowledge of the velocity of propagation of sound is of great importance in connection with acoustic signalling, especially as regards determination of range or position as in sound ranging. The velocity is very different in different substances, as it depends on the elasticity and density of the material, and therefore on its composition, pressure, and temperature. We are here concerned chiefly with the velocities in air and in sea water, although the acoustic properties of other substances require consideration when the transmitting and receiving devices are being dealt with.

For air at temperature $t°$ C. the velocity $v = 1087 + 1{\cdot}81t$ ft. per second.

For sea water $v = 4756 + 13{\cdot}8t - 0{\cdot}12t^2$ ft per second, according to the latest determination of Dr. A. B. Wood, for sea water having a salinity of 35 parts per thousand; the velocity being increased by about 3·7 ft. per second for each additional part per thousand in salinity. This gives a velocity of 1123 ft. per second in air, and 4984 ft. per second in normal sea water at a temperature of 20° C., so that the velocity in the sea is about four and a half times as great as in air.

Wave-length

As above mentioned, acoustic waves from a vibrating source onsist of a number of compressions and rarefactions following ne another and all travelling with a velocity given above. The stance between one compression or one rarefaction and the next called the wave-length of the sound, and it is evident that if the equency of vibration is n cycles per second there will be a train n waves in a distance equal to the velocity, so that the wave-ngth $\lambda = \frac{v}{n}$. For example, if we take a frequency n of 500 ~, the orresponding wave-length in air and in sea water respectively 20° C. will be:

$$\text{In air } \frac{1123}{500} = 2\cdot246 \text{ ft., and in sea water } \frac{4984}{500} = 9\cdot968 \text{ ft.}$$

The greater wave-length in water introduces somewhat serious fficulties as regards directional transmission and reception, as will ppear later

Transmission of Sound through Various Substances

As was first shown by Newton, the velocity of sound in any substance can be calculated from a knowledge of its elasticity of volume nd its density. If κ is the elasticity and ρ the density, it is easy to ove that the velocity of propagation of sound $v = \sqrt{\frac{\kappa}{\rho}}$. It must remembered, however, that with the rapid vibrations of audible unds the heating and cooling resulting from compression and refaction have no time to die away, and we must therefore take e adiabatic elasticity instead of the constant temperature or isoermal elasticity in the above formula. For gases the isothermal sticity is equal to the pressure, or about 10^6 dynes per square ntimetre in the case of ordinary atmospheric pressure, and the iabatic elasticity of air is 1·41 times this amount, while the density air $\rho = 0\cdot00129$ gm. per cubic centimetre, so that the velocity

$$= \sqrt{\frac{1\cdot41 \times 10^6}{0\cdot00129}} = 33{,}000 \text{ cm. per second, or about 1085 ft. per}$$

cond, agreeing closely with the value obtained by direct experiment.

The mathematical theory also enables us to calculate the amount

of acoustic power transmitted by sinusoidal waves, and the phenomena which result when the sound passes from one medium into another—matters of considerable importance in connection with submarine signalling. It can be shown that the relation between the pressure P due to vibration (i e. the alternating excess over the mean pressure), and the velocity V *of a moving particle*, at any point of the medium in the case of a plane wave of large area compared with the wave length, is given by the relation $P = RV$ where $R = \sqrt{\kappa\rho}$. This relation being analogous to Ohm's Law in Electricity, the quantity R has been called by Brillié the " acoustic resistance " * of the medium. The power transmitted (w) per unit area of the wave front is $\frac{1}{2}P_{max}V_{max}$, i e. $P^2_{max}/2R$, or $RV^2_{max}/2$. For a plane wave sinusoidal disturbance of frequency n periods per second, and writing ω for $2\pi n$, we have $V_{max} = \omega a$, where a is the amplitude of the displacement, so that $w = \omega a P_{max}/2 = R\omega^2a^2/2$ ergs per square centimetre per second

For ordinary sea water in which $\kappa = 2{\cdot}2 \times 10^{10}$ dynes per square centimetre, and $\rho = 1{\cdot}028$, $R = \sqrt{\kappa\rho} = 14 \times 10^4$, so that for a frequency of 500 ~ and a displacement of 0 1 mm., the power transmitted would be 7 watts per square centimetre

When sound passes from one medium into another, it can be shown that unless the two media have the same acoustic resistance there will be a certain amount of reflection at the interface If r is the ratio of the acoustic resistance of the second medium to that in the first, and the wave fronts are parallel to the interface which is large in comparison with the wave length,

$$P_2 = \frac{2r}{r+1}P_1,\ P_1' = \frac{r-1}{r+1}P_1,\ V_2 = \frac{2}{r+1}V_1, \text{ and } V_1' = \frac{r-1}{r+1}V_1$$

where P_1 is the pressure and V_1 the velocity in the original wave
P_2 " V_2 " transmitted "
P_1' " V_1' " reflected "

If the second medium is highly resistant compared with the first, so that r is very large, $P_2 = 2P_1$, $P_1' = P_1$, $V_2 = 0$, and $V_1' = V_1$ so that the pressure at the interface is double that in the original wave and the velocity, being equal to $(V_1 - V_1')$, is zero, since the movements in the direct and reflected waves are equal and in

* H. Brillié, *Le Génie Civil*, 23rd and 30th August, 1919; "Modern Marine Problems in War and Peace", 11th Kelvin lecture to Institution of Electrical Engineers, by Dr. C V Drysdale. *Jour. Inst Elect. Eng.*, 58, No. 293, July 1923, pp 591–3.

ıposite directions. The wave is therefore totally reflected back in e first medium and there is no transmission.

On the other hand, if the second medium has a very small oustic resistance compared with the first, so that r is very small,

$$P_2 = 0, \quad P_1' = -P_1, \quad V_2 = 2V_1, \quad \text{and} \quad V_1' = -V_1.$$

this case the total pressure $(P_1 + P_1')$ at the interface is zero, d the velocity $(V_1 - V_1')$ is $2V_1$, so that the velocity is doubled, ıt there is again no transmitted wave since $P_2 = 0$, and the wave totally reflected with a reversal of the velocity V_1. In the first se the surface is called a " fixed " end, and in the second a " free " ıd.

If, finally, the two media have the same acoustic resistance, so at $r = 1$,

$$P_2 = P_1, \quad P_1' = 0, \quad V_2 = V_1, \quad \text{and} \quad V_1' = 0,$$

d the wave passes on without any reflection This is the ideal ndition to be secured in transmitters and receivers

When the two acoustic resistances are not equal, it is easily shown at the ratio of the energy in the transmitted wave to that of the iginal wave, which we may call the efficiency of transmission), is $\dfrac{4r}{(r+1)^2}$, while that for the reflected wave is $\left(\dfrac{r-1}{r+1}\right)^2$. Now ı water we have seen that $R_1 = 14 \times 10^4$, and for air $R_2 = 40$ ee also table on p. 292), so that $r = \dfrac{R_2}{R_1} = 2{\cdot}86 \times 10^{-4}$, and $= \dfrac{4r}{(r+1)^2} = 0{\cdot}0011$, so that only a little over 0·1 per cent of e energy is transmitted This at once illustrates the difficulty in l underwater listening, as the sound passing through the water must nerally pass into the air before falling on the drum of the ear.

Again, the value of R for steel is about 395×10^4, so that on ıssing from water to steel $r = 28$ approximately, and the efficiency transmission is about 13 per cent, while from steel to air it is ıly 0·004 per cent. Hence for sound to pass from water through e side of a ship to the air inside, the efficiency would be only , per cent of 0·004 per cent, or 0·00052 per cent, were it not for the ct that the plates of a ship are sufficiently thin to act as diaphragm, ıd thus allow a greater transmission than if they were very thick. ı any case, however, the loss of energy is extremely great, and this ıs led to the practice of mounting inboard listening devices, either

directly on the sides of the ship or in tanks of water in contact with the hull, as will be described later.

The following table of the acoustic properties of various media has been given by Brillié.

Medium	Value of κ (kg per sq mm)	Value of ρ (C G S)	Value of $\sqrt{\frac{\kappa}{\rho}}$ (velocity metres per second)	Values of $R = \sqrt{\kappa\rho}$ in C G S units.
Steel	2×10^4	7·8	5100	395×10^4
Cast iron	0.95×10^4	7.0	3680	258×10^4
Brass	0.65×10^4	8.4	2780	234×10^4
Bronze	0.32×10^4	8.8	1910	168×10^4
Lead	0.06×10^4	11.4	725	82.5×10^4
Wood: Teak	0.16×10^4	0.86	4300	37×10^4
Wood: Fir	0.09×10^4	0.45	4470	20×10^4
Wood: Beech	0.06×10^4	0.8	2740	22×10^4
Water	2×10^2	1.0	1410	14×10^4
Rubber	Below 1 (variable according to the nature of the rubber)	1 (approximately)	Below 100	Below 1×10^4
Air	1.40×10^{-2}	0.0013	328	0.004×10^4

It should be noticed that the values of R for pine or beech wood are not greatly different from that for water, so that sound should pass from water to wood or vice versa without great reflection

Pressure and Displacement Receivers

From what has been said concerning the theory of acoustic transmission, it is evident that sound may be detected either by the variations of pressure in the medium or by the displacements they produce, in the same way as the existence of an electrical supply may be detected by the electrical pressure or by the current it produces. Acoustic receivers may therefore be classed as pressure receivers, analogous to electrical voltmeters, and displacement receivers corresponding to ammeters; but this classification is not a rigidly scientific one, as a receiver cannot be operated by pressure or by displacement alone. We have seen that the power per unit area of wave front is $\frac{1}{2}P_{max} V_{max.}$, so that unless the receiver makes use of both the pressure and velocity of displacement it receives no energy and can give no indication. A perfect pressure receiver would, in fact, constitute a fixed-end reflector, and a perfect displacement receiver a free-end reflector, in both of which cases we have seen no energy is transmitted.

The distinction between pressure and displacement receivers is,)wever, a useful one, just like that between a voltmeter and ammeter. A voltmeter is predominantly an electrical pressure-meairing device although it takes a small current, and an ammeter a irrent-measuring device although it requires a small P.D across s coils. Similarly a pressure receiver is one in which the diairagm is comparatively rigid and yields very little to the vibrations, hile a displacement receiver is one with a very yielding diaphragm. he distinction is of importance directly we consider directional ceivers, as the pressure in a uniform medium is the same in all rections while the displacements are in the line of propagation, that a pressure receiver will give no difference of intensity on eing rotated into different directions if it is so small that it does ot distort the waves, whereas a displacement receiver will give maximum when facing the source

As regards sensitiveness, however, it is evident that the best sults should be obtained when the receiver absorbs the whole the energy which falls upon it, which will only be the case when given alternating pressure on the diaphragm produces the same splacement as it does in the medium, so that the energy is mpletely transmitted into the receiving device without reflection. his will only be the case if the diaphragm is in resonance with e vibrations and the receiving mechanism absorbs so much iergy as to give critical damping

In the case of a properly designed receiver which is small comparison with the wave length, it will draw off energy om a greater area of the wave front than its own area, just as wireless aerial may absorb energy from a fairly large region ound it.

The practical construction of underwater receivers and hydrophones will be dealt with later, but it will be well at this point to ve some idea of their essential features. The simplest form of ich a receiver, which is analogous to the simple trumpet for air ception, is what is called the Broca tube, which consists of a ngth of metal tube with a diaphragm over its lower end. When is is dipped into the water, the sound from the water is comunicated through the diaphragm to the air inside the tube, and e observer listens at the free end This is moderately effective, it not very sensitive or convenient, as it makes no provision for nplifying the sound, and it is not easy to listen through long bent bes, so that the observer must generally listen only a few feet

above the water. Modern hydrophones are therefore nearly all of an electrical character, containing microphones or magnetophones from which electrical connections are taken to ordinary telephone receivers at the listening point.

Microphones are generally used, as they are more sensitive, and there are two types of microphone which correspond approximately to pressure or displacement receivers respectively. The former is termed the "solid back" type, in which a number of carbon granules are enclosed between a metal or carbon plate forming or attached to a diaphragm and a solid fixed block of carbon at the back. If pressure is applied to the diaphragm it compresses the granules and increases their conductivity, so that a greater current passes from a battery through the microphone and the receivers and reproduces the sound through the pressure variations. In the "button" type of microphone, on the other hand, the carbon granules are enclosed in a light metallic box or capsule covered by a small diaphragm, and the whole arrangement is mounted on a larger diaphragm, so that its vibrations move the capsule as a whole and shake up the granules, with only such changes of pressure as result from the inertia of the capsule. In this case it is the motion or displacement of the diaphragm which produces the variations of resistance in the microphone.

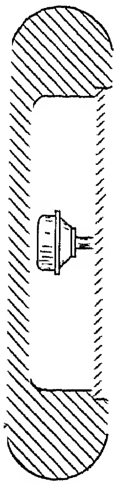


Fig 1.—Non-directional Hydrophone

The commonest type of simple hydrophone is diagrammatically shown in fig. 1 and illustrated in fig 18 It consists simply of a heavy circular metal case of disc form with a hollow space covered by a metal diaphragm to the centre of which a button microphone is attached. It is fairly sensitive but has no directional properties.

Directional Transmission and Reception

The problems of directional transmission and reception are among the most important as regards acoustic transmission. As in the case of wireless telegraphy or telephony, acoustic transmission suffers greatly from the difficulty that sound, like wireless waves, tends to radiate more or less uniformly in all directions, with the

esult that its intensity rapidly diminishes according to the inverse quare law, and there is great difficulty as regards interference and 'ant of secrecy. Again, as regards reception, it is of comparatively ttle value to have a sensitive receiver which will detect the existence f a source of sound at a great distance if it gives no indication of ne direction or position of the source. On this account the ques-on of directional transmission and reception is of at least equal npoi tance to that of power-ıl transmitters and sensitive ceivers. This question of irectional transmission and ception has received a large nount of attention.

The Binaural Method f Directional Listening. - Our own ears form a ery efficient directional re-eiving system When a idden noise occurs we istinctively turn towards ie source, and if we are lindfolded we can gene-lly tell with considerable curacy the direction from hich a sound comes. This due to the fact that, as ır two ears are on op-osite sides of the head id about 6 in. apart, the und reaches one ear a tle sooner than the other, ıless it arises from a point in a plane perpendicular to the ie joining the ears, i.e. directly in front of, behind, or above our ad. Our ears are exceedingly sensitive to this minute differ-ce of time, and as this interval depends upon the direction, tting larger the more the source is on either side, we learn to timate the direction fairly closely, provided our two ears are nearly ually sensitive. This is known as the binaural (two-ear) method estimating direction, and it has been developed both for air and bmarine listening For example, if we take two trumpets fixed a horizontal bar (fig. 2), each of which is provided with a definite

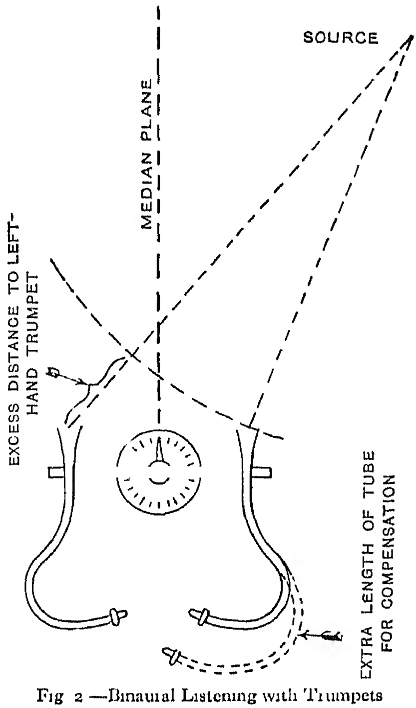


Fig 2 —Binaural Listening with Trumpets

length of rubber tubing to an ear piece, we can detect and locate an aeroplane with considerable accuracy from the noise of its engines, as the trumpets magnify the sound, and the sensitiveness to direction may be increased by increasing the distance between the trumpets. When a sound is heard, the observer swings the bar carrying the trumpets round in the direction indicated, and as he does so the sound appears to cross over from one ear to the other behind his head. The position at which this occurs is called that of binaural balance, and when this balance is obtained the bar is at right angles to the direction of the source.

The same principle can obviously be applied to underwater listening with two receivers, but in this case it should be noted that, as the velocity of sound in water is about four and a half times that in air, the distance between the receivers must be increased in that proportion to obtain the same difference of time, and therefore equal binaural discrimination As this involves the use of a somewhat long bar, which is troublesome to turn under water, recourse is generally had to what is called a binaural compensator for determining the direction.

Returning to our pair of trumpets in fig. 2, suppose that the source of sound is to the right of the median plane, and that the tubes from the trumpets, instead of being of equal length, are of different lengths, so that the additional length of tube to the right-hand trumpet is equal to the extra distance from the source to the left-hand trumpet. In this case it is evident that the delay of the sound in reaching the left-hand trumpet is balanced by the extra delay between the right-hand trumpet and the ear, and that binaural balance will be obtained although the source is on one side of the median plane. It is therefore possible to obtain the direction of a source with a fixed bar carrying the receivers, provided that arrangements can be made for varying the length of the stethoscope or ear tubes, and such an arrangement is called a binaural compensator, the most simple form of which is shown in fig. 3. Here the two equal tubes from the trumpets are brought to the two ends of a long straight tube, which is, however, made of three sections, the middle one sliding in the two end portions. The middle tube is blocked at its centre, and is provided with two apertures from which two equal rubber tubes are taken to the ear pieces. When the centre section of tube is in its middle position the two lengths of air path from the trumpets to the ear pieces are the same, and binaural balance will therefore only be obtained when the source is

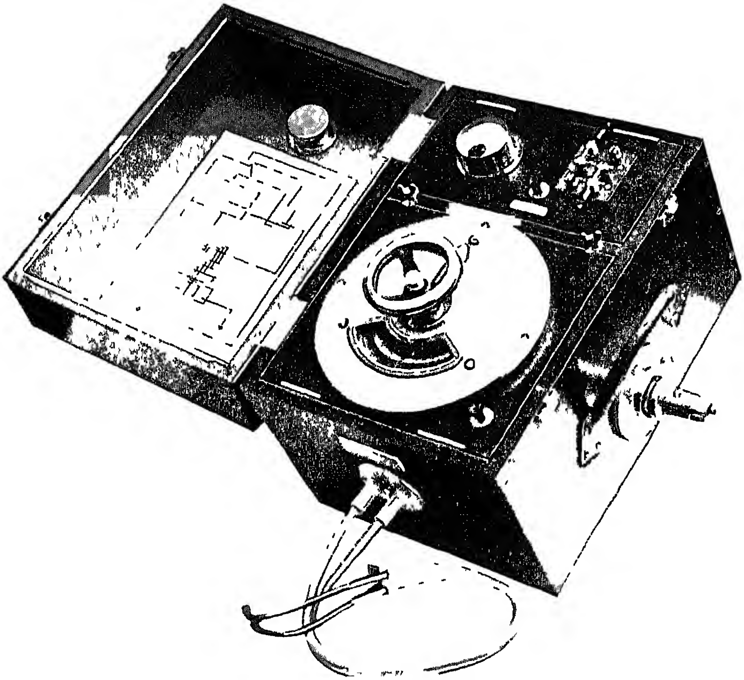


FIG. 4(a).—AMERICAN COMPENSATOR FOR DIRECTIONAL LISTENING

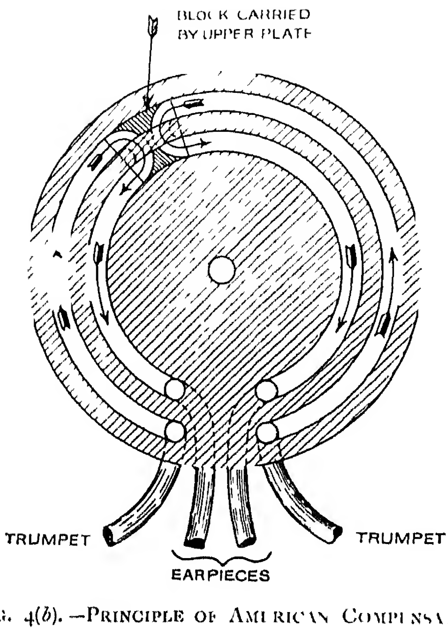


FIG. 4(b).—PRINCIPLE OF AMERICAN COMPENSATOR

he median plane; but if the source is to the right of this plane, that the sound reaches the right-hand trumpet first, sliding the tre tube to the left increases the path from the right-hand npet and diminishes that from the left-hand one, so that balance be restored, and an index on the sliding tube will read off the ection on a suitably engraved scale which can be divided from relation $2d = b \sin\theta$, or $\sin\theta = \frac{2d}{b}$, where b is the distance between trumpets, d the displacement of the central tube from its mid ition, and θ the angle obliquity of the direc- ı of the source from median plane.

In order to carry out ectional listening on se lines with the atest convenience, a ular form of compen- ɔı has been designed he United States and de by the Automatic ephone Company 4*a* shows the ex- ıal appearance of this npensator, and fig. 4*b* essential feature of its construction Two concentric ular grooves are cut in a fixed plate, and are covered by a te which converts them practically into circular tubes This te can be rotated above the fixed plate, and is provided with) projections which close the grooves but connect the inner and er ones together by two cross channels. The sound from the) trumpets, entering the two ends of the outer groove, travels ınd this groove to the stop and then passes through the ınnels to the inner grooves, returning to its two ends, to which : ear pieces are connected. It is evident that as the upper te is turned the difference of path between the two systems altered by four times the distance through which the stop vels, and a pointer on the top plate indicates the direction on dial.

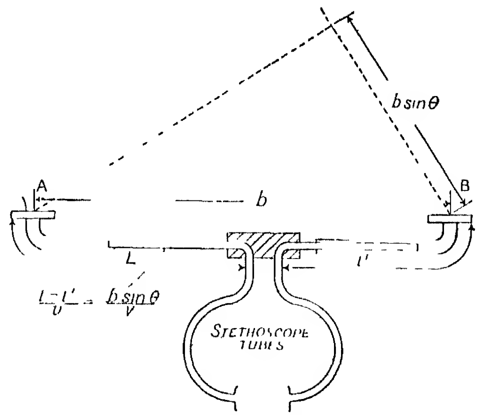
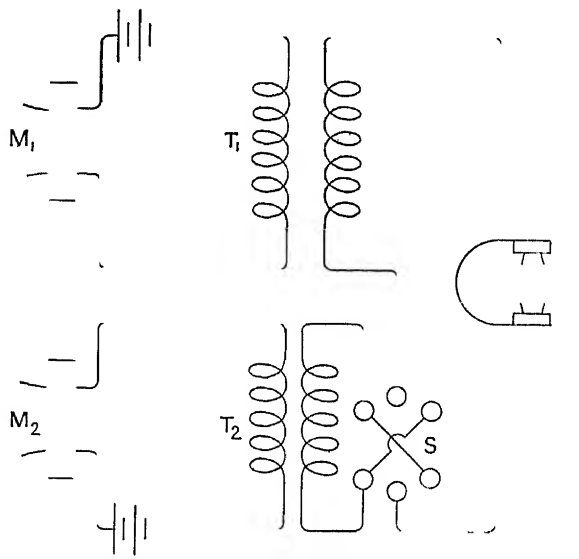


Fig 3 —Binaural Method with Rectilineal Compensator

This binaural principle is of such importance that it has been scribed at length, and many applications of it will be seen later,

but there are other methods of directional reception which may first be referred to.

Sum-and-difference Method.—In the case of electrical receivers the binaural method may be replaced by what is called the sum-and-difference method. Suppose, in fig. 5, that our two trumpets on the bar are replaced by two similar ordinary microphone receivers M_1 and M_2, and that these receivers are connected to two telephone transformers T_1 and T_2, the secondaries of which can be



SUM-AND-DIFFERENCE METHOD

Fig. 5

connected in series as shown. It is evident that if the source is in the median plane, so that the sound reaches both receivers simultaneously, they should be similarly affected and produce equal electromotive forces in the transformer secondaries. If these secondaries are connected so as to assist one another, a loud sound should be heard, but if one of them is reversed by the switch s the two electromotive forces should be equal and opposite, and silence should result. But if this is the case, and the source moves to one or other side of the median plane, the sound will reach the two receivers at different times, and cancellation should no longer take place, so that the source should appear louder the greater the angle of reception from the median plane. By swinging the bar

til the sound vanishes, or at least becomes a minimum, the direc-n of the source is given just as in the binaural method, and in s position the sound will be a maximum when the transformers ıist one another.

This sum-and-difference method has the advantage over the ıaural method that it does not depend on the binaural sensitive-ss of the observer, which may be very poor, especially in the case peisons with partial deafness in one ear; and on this account ne observers prefer it On the other hand, it reintroduces the jectionable feature of swinging the bar unless a compensator is roduced between two sets of receivers which introduces undesir-le complication. But in any case this sum-and-difference method of great value in connection with electrical receivers, as it brings t a difficulty which has to be overcome before such receivers can used for binaural listening It will be noticed that for silence be obtained with the difference connection the sound must affect th receivers equally, but this is very rarely the case with ordinary crophones, owing to differences in the properties of their dia-ragms. In fact, if two such receivers are placed close together as to receive the same sound, it is not uncommon to find very le difference between the sound heard with the sum-and-difference ınections, and in this case such receivers are quite useless for ıaural listening, which depends upon perfect similarity of response. replacing the ordinary metal or carbon diaphragms by rubber mbranes, however, much greater equality can be secured, and the n-and-difference method can be used in the test room to test s equality and to select perfectly paired receivers either for binaural for sum-and-difference direction finding

Directional Receivers.—It has already been pointed out t although the pressure changes in an acoustic beam have no ection, the displacements take place in the direction of propaga-n, and that a displacement receiver should therefore have direc-ıal properties. This principle has not actually been employed directional listening to any extent, but Mr. B. S. Smith has ısed a displacement receiver consisting of a small hollow sphere ıtaining a magnetophone transmitter, the whole arrangement ng of neutral buoyancy. Such a sphere vibrates as if it were t of the water, and consequently gives maximum effect on the gnetophone when its axis is in the direction of propagation and o when it is perpendicular to it.

The type of directional receiver which has been most employed

in practice, however, is of a balanced type, as shown in figs 6 and 7. It is similar to the non-directional hydrophone (fig. 1), except that instead of a thick hollow metal

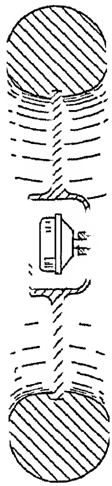


Fig 6 —Bi-directional Hydrophone

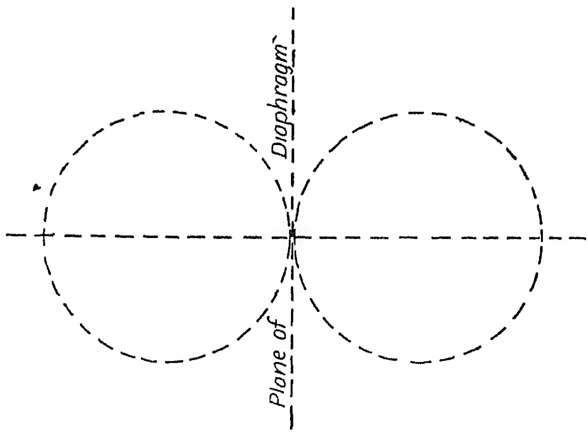


Fig 7 —Polar Curves of Intensity for Bi-directional Hydrophone

case it has simply a heavy brass ring with a central diaphragm having a hollow boss at its centre in which the button microphone is fixed. Obviously if such an arrangement is placed so that its plane lies along the direction of propagation, the pressure falls upon both faces of the diaphragm equally and simultaneously and no motion results, so that nothing can be heard in this position. When the hydrophone is turned with one of its faces towards the source, however, the back face is screened by the ring, and the sound reaches it later and with less intensity, so that there is a resultant effect. On turning such a hydrophone round, therefore, the sound is a minimum when the edge points towards the source, and rises to a maximum when turned through a right angle, the intensity for various angles of turning being shown in the polar diagram fig. 7. A similar effect is given by the Morris-Sykes directional hydrophone (fig. 8), which has two similar diaphragms on its two faces connected by a rod at their centres, on which the microphone is mounted. As the variations in pressure tend to move the two

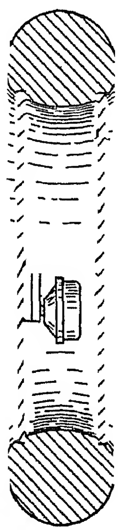


Fig 8 —Morris-Sykes Hydrophone

phragms in opposite directions, no movement of the bar is)duced and no sound heard when the hydrophone is edge on. These forms of directional hydrophone are fairly effective, giving iily sharp minimum, but they do not entirely fill the requirements directionality, as it is evident that minimum is given when either ;e of the disc points to the source, so that the source may be in ier of two diametrically opposite directions. For this reason they called bi-directional hydrophones; but it has been found possible get over this difficulty and to convert a bi-directional into a uni-

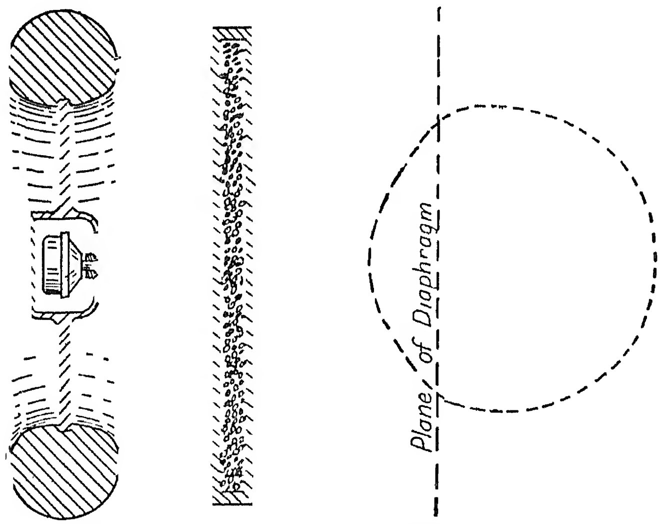


Fig 9 —Uni-directional Hydrophone and Polar Curve of Intensity

'ctional hydrophone, by simply mounting what is called a iffle plate " a few inches away from one face, as shown in fig. 9. s baffle plate may be made of layers of wood or metal or have ivity filled with shot in it, so that it tends to shield the sound n one face. Such a hydrophone gives the loudest sound when unbaffled face is turned towards the source and the weakest nd when it is turned directly away from it, the intensity in ous directions being shown by the polar curve, so that there ow no ambiguity as to direction, and the device is then called ni-directional hydrophone. It does not, however, give such nite indications of direction as the sharp minima of the bi-direc- al form, and it is therefore better to couple a uni-directional a bi-directional hydrophone at right angles to one another on

the same vertical shaft. When maximum intensity is observed on the former and a minimum on the latter, the direction of the source is definitely given.

Besides the foregoing methods of directional reception there are others, such as those of Professors Mason and Pierce, depending on the principle of acoustic integration first enunciated by Professor A. W. Porter, which leads to the use of large flat surfaces for reception, and the Walser gear in which the sound is brought to a focus by a lenticular device, as will be described below.

As regards directional transmission, it may first be mentioned as a general principle of all radiation that transmission and reception are reciprocal problems, that good receivers make good transmitters, and that a directional receiver will make a directional transmitter with the same distribution of intensity in different directions. For example, if, instead of listening by means of two trumpets coupled by equal tubes to the ear, we bring the two tubes to a powerful source of sound so that the sound escapes in an exactly similar manner from the two trumpets, an observer in median plane will hear this sound very loudly, but as he moves to one or other side of this plane the sound will appear fainter. Similarly, by vibrating the diaphragm of a uni-directional hydrophone sound will be emitted chiefly in one direction, and by extension of this principle a beam of sound may be sent in any direction we please

PRACTICAL UNDERWATER TRANSMITTERS AND RECEIVERS

We can now turn to the actual devices employed for submarine signalling, and they may be described under the headings (*a*) transmitters, (*b*) receivers, and (*c*) directional devices.

Submarine Transmitters or Sources of Sound

The simplest form of submarine transmitter is the submarine bell which has been used as an aid to navigation for many years. Originally suggested by Mr. Henry Edmunds in 1878, it was not until 1898 that it was taken up seriously as a practical navigational device by Mr. A. J. Munday and Professor Elisha Gray, who formed the Gray Telephone Company in 1899, and employed a bell struck under water with a submerged telephone receiver. After Professor Gray's death in 1901, the work was carried on by Mr

ıday, who staıted the Submaıine Signal Company to take over ›perations. Various forms of submarine bell were expeıımented , but the form which was finally adopted is shown ın fig. 10, :onsists of a bronze bell, weighing 220 lb. and having a frequency ;15 ~ in water, which is struck by a hammer generally operated ompressed aır. A twin hose pipe is used to supply the com- ıed aır and to convey away the exhaust aır from the appa- , and the strokes are ıegulated by ode valve", which consists of a [dıaphragm actuating the main ıpply to the hammeı mechanism type of bell ıs generally used on ships, ın which case it is sımply overboard to a depth of 18 to ., but ın the case oi lıghthouses e electric supply ıs avaılable an ically operated bell oi the same is employed, which ıs hung on ›od stand about 25 ft. hıgh and . spread, standıng on the bottom ıy convenıent posıtıon up to a or so from the lıghthouse. In :ase the hammer ıs opeıated by :ulaı ıron armatuıe attıacted to ectıomagnets on a common yoke, ole faces beıng coveıed by coppeı to prevent stıckıng by ıesıdual etızatıon A fouı-coıe cable ıs ded, two for supplyıng the $3\frac{1}{2}$ of operatıng cuırent, the otheı ›eıng connected to a telephone nitter ın the mechanısm case, which enables the operatoı to f the bell ıs workıng propeıly. The first of these electııcally ted bells was laıd down at Egg Rock, near Boston Haıbour, d States, and a large number of pneumatically and electrically ed bells aıe now in service round the British and American .

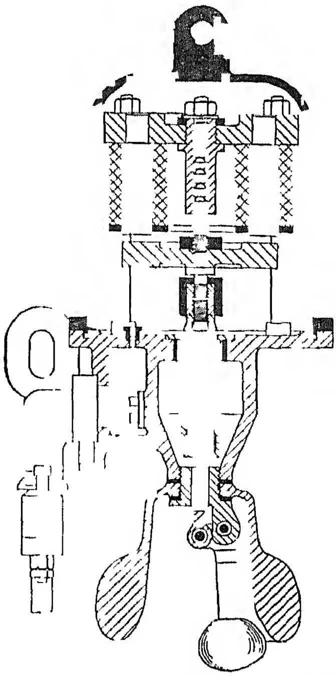


Fig 10—Submaıme Signal Company's Bell Electııcally operated type

Electromagnetic Transmitters

On account of the ease of the operation and control, electromagnetic transmitters have been most popular, and they are now made up to large sizes transmitting hundreds of watts of acoustic power. They may be divided into two classes: (*a*) continuous, and (*b*) intermittent or impulse transmitters

(*a*) **Continuous Electromagnetic Transmitters.**—In all these transmitters alternating current is employed, of frequency corresponding to the natural vibration frequency of the vibrating system, and this current may be used either to energize a laminated electromagnet which acts on the diaphragm, or to traverse a coil in a powerful steady magnetic field, thus developing an alternating force which can be applied to the diaphragm. These two types of transmitter may be called the "soft-iron" and the "moving-coil" types respectively. In the former type the frequency of the note is double that of the alternating current as the diaphragm is attracted equally when the current flows in either direction, but in the latter type the note frequency is the same as that of the current.

The soft-iron type of continuous transmitter has been greatly developed by the Germans, and fig. 11 shows one of the most generally used types constructed by the Signalgesellschaft of Kiel. The diaphragm D is provided with a boss at its centre, to which is fixed a casting carrying a laminated E-shaped iron core C nearly in contact with a similar block of stampings C′ above The exciting coil encircles the inner pole of these stampings, as in the familiar core type of transformer, and produces a powerful attractive force at each passage of the current in either direction, so that the frequency of variation of the force is double that of the current The upper block of stampings is not rigidly fixed, but is coupled to the lower block through the agency of four vertical steel tubes T with steel rods inside them, the lengths of these rods and tubes being such that the natural frequency of their longitudinal vibrations is equal to that of the diaphragm. The diaphragm is bolted to a conical housing with glands for the introduction of the supply cables. A transmitter of this type, having a total weight of about 5 cwt. and a diaphragm about 18 in. diameter, gives an acoustic radiation of 300 to 400 watts, the mechanical efficiency being about 50 per cent.

A great objection to these moving iron transmitters is their inherently low power factor owing to their great inductance, which involves a large wattless exciting current. This can, of course, be

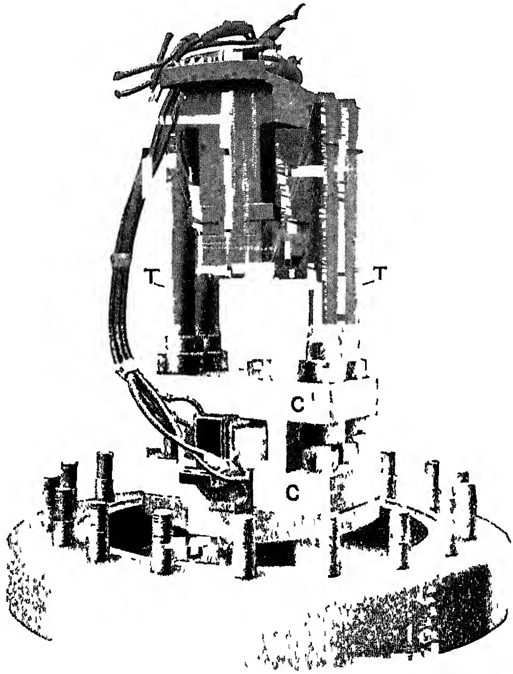


FIG. 19. CONTINUOUS ELECTROMAGNETIC TRANSMITTER AS CONSTRUCTED BY THE SIGNALGESELLSCHAFT OF KIEL.

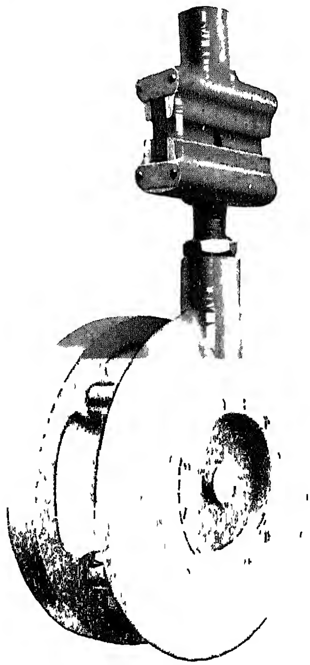


FIG. 20. SINGLE DIAPHRAGM BI-DIRECTIONAL HYDROPHONE CONVERTED INTO UNI-DIRECTIONAL INSTRUMENT BY ADDITION OF BAFFLE PLATE.

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tified by using a large condenser in parallel or series with the iting coils, but this is not a very satisfactory expedient.

On this account the moving-coil type of transmitter has been oured, especially by the ıericans, and its fundantal principle is diagramtically shown in fig. 12. ıe coil of wire traversed the alternating current is ached directly to the diaragm, and moves in the nular field of a powerful ɔot magnet" excited by ·ect current. This type s relatively little inducıce, and therefore a high ıwer-factor, but its conruction is mechanically difficult, as the coils of wire do not rm a rigid mass and are therefore liable to cause great damping ıd loss of efficiency.

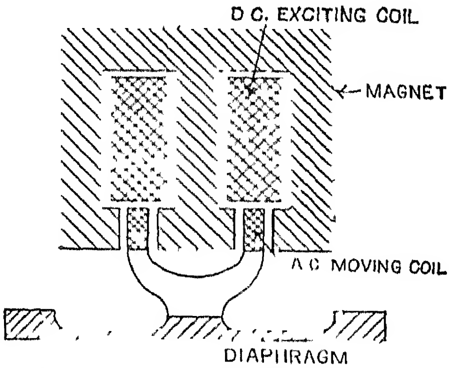

Fig 12.—Principle of Moving-coil Transmitter

This difficulty was very neatly got over by Fessenden in the nited States, and the Fessenden ansmitter is probably the most ficient and powerful of all electroıagnetic transmitters The prinple is exactly the same as above, ut, instead of mounting the coil irectly on the diaphragm so as to ıove with it, Fessenden employs a xed coil which induces currents in copper cylinder by transformer ction, and this copper cylinder is ttached to the diaphragm. Fig. 13 hows a diagrammatic section of a 'essenden transmitter in which the ırect-current electromagnet is biıolar and encircles the copper cylinler which is attached to the diaıhragm. The alternating current traverses a fixed coil wound on an ıner iron core, the coil being wound in two halves in opposite direcions to correspond with the two poles of the magnet, and this coil

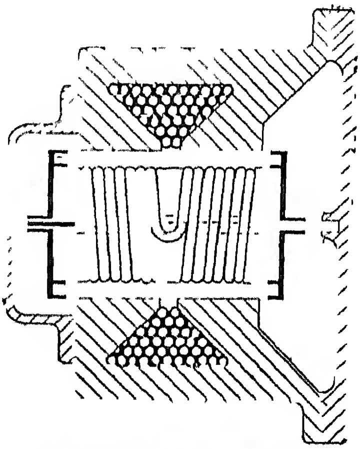
Fig. 13.—Diagram of Fessenden Transmitter

induces powerful currents in the copper cylinder which traverse the strong field of the magnet and impart longitudinal forces to it of the same frequency as that of the alternating current. The arrangement is therefore very rigid mechanically, and a high power-factor and efficiency are obtained at resonance, which is usually for a frequency of 500 ~. Transmitters of this type giving an acoustic radiation of 500 watts or more have been constructed, and are capable of signalling under water to a distance of 300 miles or thereabout Morse signals can be sent by either of the above types of transmitter by the aid of a suitable signalling key.

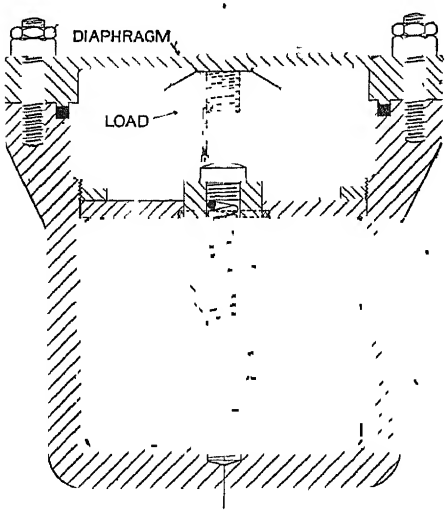

Fig 14.—Diaphragm Sounder

(*b*) **Intermittent or Impulse Transmitters.**—Reference has already been made to the submarine bell, which was the first type of intermittent submarine transmitter and which can be operated electromagnetically A more simple type of impulse transmitter is the diaphragm sounder of Mr B. S Smith, which has the advantage over the bell in that the striking mechanism is totally enclosed and therefore does not work in water. Fig. 14 shows a section of a sounder of this type, which is provided with an ordinary steel diaphragm with centre boss against which a cylindrical hammer strikes. This hammer is withdrawn on passing direct current through the exciting coil, against the force of a spiral spring, and upon the sudden interruption of the current the spring causes the hammer to strike the diaphragm with a single sharp blow thereupon rebounding and leaving the diaphragm free to vibrate A very powerful impulse, though of brief duration, owing to the heavy damping of the water, is produced in this way.

Similar powerful impulse transmitters have been constructed in which the hammer is operated pneumatically by compressed air at a frequency of about a hundred blows per second, and this type of

ısmıttei can be used for sıgnallıng in the Morse code, by means a suıtable pneumatic key.

A sımple tıansmitter has been specıally designed by the hor for acoustıc depth soundıng, the object being to gıve a ies of single impulses to the water wıthout vibration. Here the stıc diaphragm ıs entirely done away wıth and ıts place taken a square lamınated plate, whıch is attracted to an E-formed ıinated magnet on passing dırect current round an excıting coil ircling the centre pole. The attıactıve force is $\frac{B^2}{8\pi}$ dynes per ıare centımetre, where B ıs the magnetic field ın gausses, so that B = 15,000, the foıce ıs about 14 Kgm. peı square centimetıe, l a pole area of 140 sq. cm. gives a total foıce of about 2 tons. order to ımpart this foıce to the water, the pole faces and plate grooved, and ındıa-rubber stııps ınseıted whıch are compressed the attractıon of the magnet On switchıng on this tıansmıtter a 100-volt cıı cuit the current ıises compaıatıvely slowly, owıng ıts great ınductance, and the plate ıs gıadually drawn up, but on ldenly bıeakıng the curıent the reactıon of the ıubbeı strıps ıots the plate suddenly forward wıth an ınıtıal foıce of about ons, and ımparts a sıngle sudden shock lıke an explosıon to the ter The use of thıs transmıtteı wıll be explaıned ın connectıon h acoustıc depth soundıng

Submarine Sirens

A number of forms of submaıınc sıren, ın whıch plates oı cylin-s provıded wıth holes thıough whıch jets of water pass when the tes oı cylındeıs aıe rotated, have been devısed both ın thıs country l ın Germany, and are extremely poweıful By suitably bevellıng holes, the wateı pıessure can, of course, be made to ıotate the tes, but this is objectıonable fıom the sıgnallıng poınt of view, it involves a gradual ıunning up to speed and a consequent iatıon ın the frequency of the note On thıs account the plate cylinder is usually rotated ındependently at a constant speed by electrıc motor, and signalling ıs effected by swıtchıng on and the hıgh-pressure water supply. These sirens have not, how-r, come greatly into use, as the electromagnetic transmitters are much more convenient, and they will therefore not be described detail.

There are many other forms of acoustic transmitters, but the

above are most generally useful for acoustic signalling or impuls transmission. For sound-ranging purposes small explosive charge are sometimes employed.

Receivers or Hydrophones

The C Tube

The earliest and most simple of all subaqueous acoustic receiver as already mentioned was the Broca tube, consisting of a length of meta tube with a diaphragm stretched ove its lower end The Americans have improved this form of tube, by re placing the diaphragm by a thick-walled rubber bulb or teat, and have called it the C tube (fig 15) from Dr. Coolidge, its inventor It is fairly sensitive, but the amount of energy communicated to the air within the bulb is very small by the principle of transmission given above, and it suffers from the inconvenience of requiring the observer to listen at the end of a somewhat short tube.

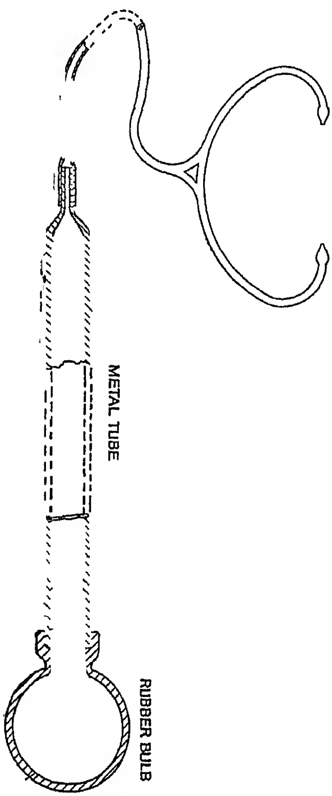

Fig 15.—C Tube

The advantages, as regards sensitiveness and convenience, of employing microphones were also appreciated by the Americans who enclosed microphones in hollow rubber bodies, and a combination of three such bodies was often floated on a triangular frame and employed for binaural listening Fig. 16 shows a double C tube arrangement for binaural listening. As has already been explained, binaural listening on two receivers permits the direction of the source to be ascertained, but it is evident that the direction suffers from the same ambiguity as in the bi-directional hydrophone, as a source symmetrically situated on the other side of the line joining

two hydrophones would give the same difference of time of ral. By using three hydrophones arranged at the corners of equilateral triangle, and uralling on each pair in , this ambiguity disaps. The necessity for ectly pairing the microaes by the sum-and-difnce method has been idy referred to.

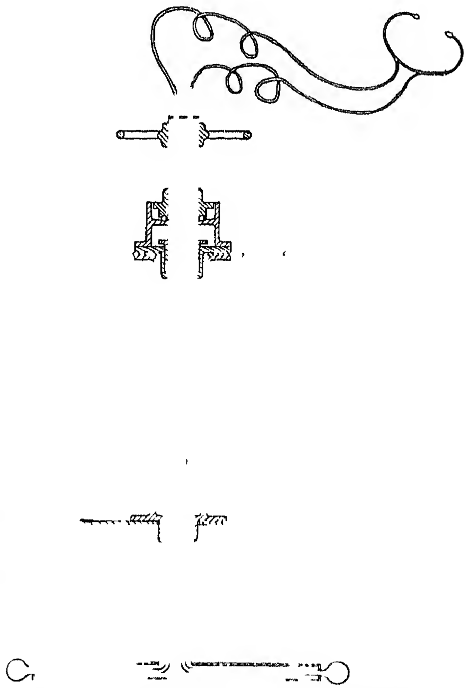

Fig 16 —Double C Tube Binaural Arrangement

Magnetophones

Although greatly inferior sensitiveness to microones, magnetophones have e advantages for under-r listening, as they are from the vagaries of ular microphones and be more easily paired binauralling As their itiveness can be enhanced most any extent by the ern valve amplifiers, h cannot be employed microphones owing to the grating or "frying" noise prod by the granules, they can be made equally effective.

The Fessenden transmitter described on p. 305 can be used as a erful magnetophone receiver by exciting its magnet and listening he coils, which are supplied with alternating current when mitting, and it is commonly used as a receiver in signalling, as of course, in tune with the note of all such transmitters This) tuning, however, renders it unsuitable for general listening oses.

ne of the most effective magnetophone devices for inboard ing is the "air-drive" magnetophone of Mr. B. S. Smith (7). It consists of a massive lead casing (4) fixed to the side of aip, carrying a thick india-rubber diaphragm (2) in contact with vater. Close behind this diaphragm an ordinary Brown reedtelephone receiver (3) is mounted, so that the sound transmitted

from the water to the air behind it causes the diaphragm and reed of the receiver to vibrate and induces currents in the receiver windings This type of receiver connected to a three-valve amplifier and high-resistance telephones gives a fairly faithful reproduction of ordinary sounds; and if four of these receivers are mounted on the hull in positions fore and aft and port and starboard, the screening effect of the hull enables the direction of the source to be estimated from the relative intensities on the four receivers—a four-way change-over switch being interposed between the receivers and the amplifier. Ship noises are greatly diminished by fixing the lead

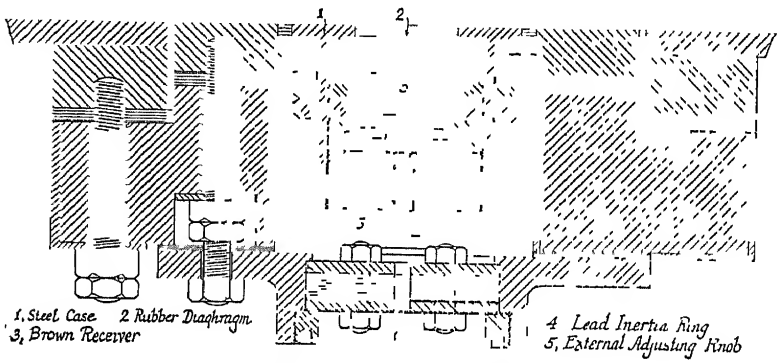

Fig 17 —Air-drive Magnetophone

ring to the plates with a rubber seating, as the great inertia of the lead (4) prevents it from taking up the hull vibrations readily.

There are many other forms of receivers, but the above are the principal ones which have been used for underwater acoustic reception.

Practical Construction of Hydrophones

A few illustrations may now be given of the actual forms of some of the most generally used hydrophones. Fig. 18 shows the simplest form of non-directional hydrophone, of which a diagram was given in fig. 1, in which a heavy hollow bronze casting is provided with a diaphragm on one side, to the centre of which a small "solid back" microphone is attached.

Fig. 19 is an illustration of the double-diaphragm bi-directional hydrophone, diagrammatically shown in fig. 8, and fig. 20 (see plate facing p. 316) shows a single-diaphragm bi-directional hydro-

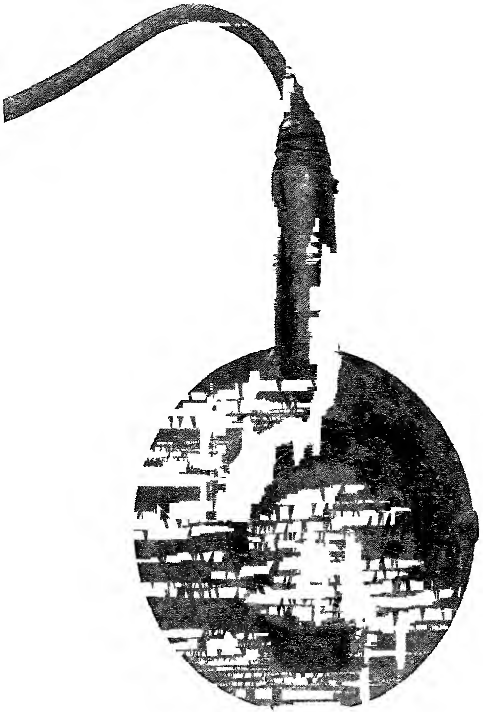

FIG. 18.—NON-DIRECTIONAL HYDROPHONE

FIG. 19.—DOUBLE-DIAPHRAGM BI-DIRECTIONAL HYDROPHONE

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ɪhone converted ınto a unı-directional instrument by the addition ɪf a baffle plate, as ın fig. 9.

In order to be able to listen from a ship in motion and to reduce hip and water noıses as much as possible, hydrophones, either of he rubber-block form or of one of the foregoıng types, have been nclosed in fish-shaped bodies and towed through the water some lıstance astern, and combinations of such bodıes have been used or directıonal listening by bınaurallıng. The modern tendency,

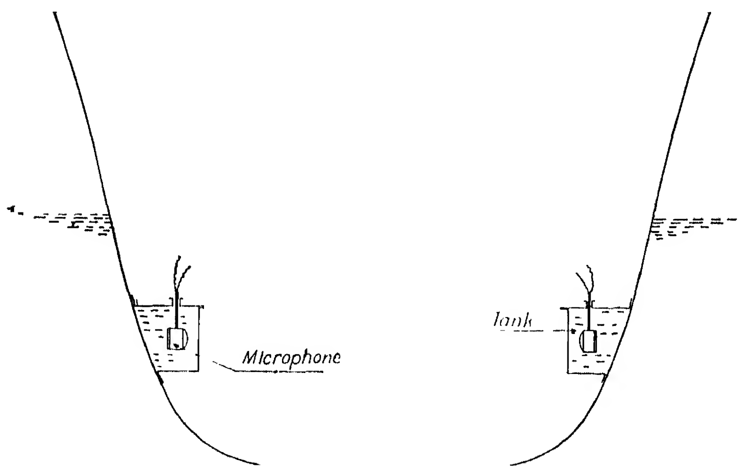

Fıg 21 —Reception by Hydrophone ın Tanks

owever, has been ın the directıon of ınboard listening, by securıng ficıent acoustıc ınsulatıon from the hull.

The method of listenıng in tanks ınsıde the hull, fiıst intro-ıced by the Submarine Sıgnal Company, has been greatly lopted by the Germans. Fıg. 21 shows the dısposıtıon of a paır these tanks with the hydrophones insıde. This devıce avoıds ıe great loss by reflection on passing from water to air, as has ɜen referred to above.

A remarkably ınteresting and effectıve form of directional ın-ɔard lıstenıng device, however, is that known as the Walser gear, ɜvised by Lieutenant Walser of the French navy, in which the

sound is brought to a focus, as in a camera obscura, and the direction determined by the position of this focus For this purpose a "blister", consisting of a steel dome A of spherical curvature and about 3 ft. 6 in. diameter, part of which is seen in fig 22, is fitted to the hull, and this steel dome is provided with a large number of apertures B into which thin steel diaphragms C are inserted. These

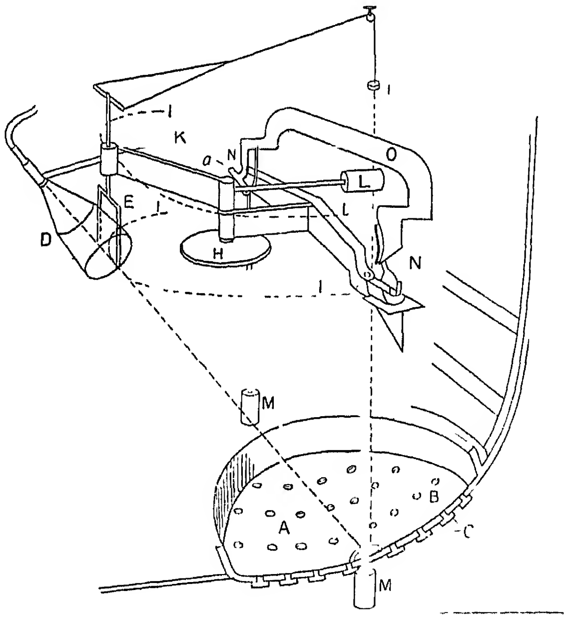

Fig 22 —Walser Apparatus

diaphragms being on the spherical dome collect the sound and direct it to a focus at a distance of 5 or 6 ft A trumpet D, to which a stethoscope tube is attached, is mounted on an arm E turning on a vertical axis, so as to be able to follow the focus and point in the direction of the sound from whatever direction it comes. Two of these blisters are generally mounted somewhat forward on the two sides of the hull, and an observer seated between them applies the tubes from the two trumpets to his ears, so that he can follow the position of the source on either side, the direction being given on a scale when the maximum intensity is obtained.

Directional Devices

Sound Ranging

One of the most important acoustic applications in the War 'as that of sound ranging for the detection of the position both of uns and of submarine explosions, the importance of which is bvious. There are two chief methods of location, which may be escribed as "multiple-station" and "wireless-acoustic" sound inging respectively, but the former, although less convenient, was ie only one employed in the War, as it needs no co-opeiation on ie part of the sending station.

Multiple-station Ranging.—The multiple-station method

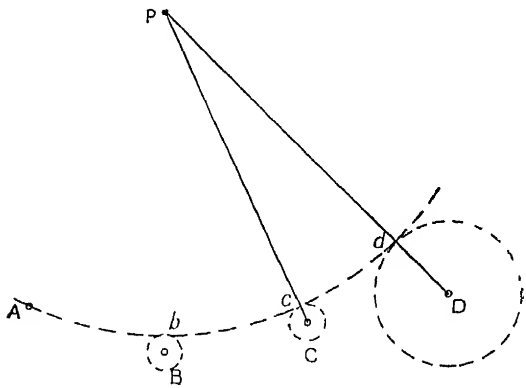

Fig 23 —Sound-ranging Diagram

sound ranging depends on the principle that sound waves are at out as spheres with centre at the source of sound. If three more receivers are therefore set up on a circle with centre at : source, the sound will arrive at all of them simultaneously, so it if the signals are all coincident the source must be at the centre the circle passing through the receivers. If, however, the source in any other position the signals will be received at different times, d if the differences of the times of reception are measured the sition of the source can be located by calculation, or graphically.

A simple diagram (fig. 23) will make this method clear. Let 3CD be four receivers in any accurately known positions and pe the position of an explosion to be located. If we draw a circle h P as centie through the receiver A, it is evident that when the ind arrives at A it still has the distances *b*B to travel before arriving

at B, and cC and dD before arriving at C and D respectively, so that the times of arrival at B, C, and D are $t_1 = \frac{bB}{v}$, $t_2 = \frac{cC}{v}$, and $t_3 = \frac{dD}{v}$ behind that at A. Consequently if we can measure the time intervals t_1, t_2, and t_3, and multiply them by the velocity, we get the perpendicular distances of the station B, C, and D from the circle passing through the source, and if we draw circles round B, C, and D with radii to scale representing these distances, the centre of a circle tangential to these circles will be the position of the source P.

The method of determining these time differences almost entirely employed during the War was by means of a multiple-stringed Einthoven galvanometer, four of these strings being connected to four microphones or hydrophones, while a fifth was connected to an electric clock or tuning fork, so as to give an accurate time scale. The image of the strings was focused on a continuous band of bromide paper, which was drawn through the camera and a developing and fixing bath by means of a motor, so that it emerged from the apparatus ready for washing and drying, though the times could be read off instantly it appeared. To facilitate the reading off of the time intervals, a wheel with thick and thin spokes was kept revolving in front of the source of light by means of a "phonic motor" in sychronism with a tuning fork, so that a number of lines were marked across the paper at intervals of hundredths and tenths of a second.

Fig. 24 is a reproduction of a sound-ranging record so obtained, on which the times of reception at four receivers are marked, and fig 25 a view of the Einthoven camera outfit employed The receivers used in this case were simple microphones, mounted on diaphragms bolted on watertight cases mounted on tripods lowered on the sea bottom and accurately surveyed, the microphones being connected by cables to the observing station

On account of the importance of sound ranging as a means of locating the position of a ship in a fog, efforts have been made to improve it still further, and to eliminate the photographic apparatus. The greatest achievements in this direction have been made by Dr. A. B. Wood and Mr. J. M. Ford at the Admiralty Experimental Station, who have devised what they call a "phonic chronometer" for indicating the time intervals directly on dials to an accuracy within one-thousandth of a second. The principle of the instrument is very simple, and can readily be understood by reference to fig. 26. A phonic motor with vertical spindle revolves with a

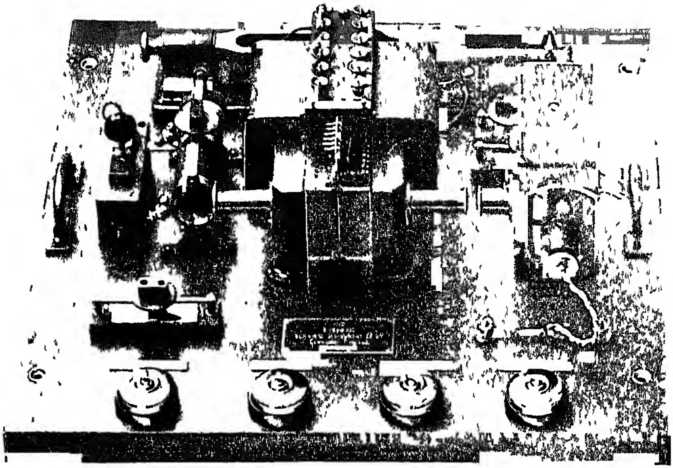

FIG. 25—EINTHOVEN CAMERA FOR SOUND RANGING

FIG. 26—THREE DIAL PHONIC CHRONOMETER

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sist in this case of diaphragms with single-point contacts which a thrown off on arrival of the shock, and remain broken until the are restored by electromagnets. Each of these contacts is connecte to the electromagnet windings of the dials as shown, and it will l seen that as each contact is broken it breaks one of the circuits i either one or two of the dial mechanisms, and starts the pointe revolving until the breaking of another contact breaks the secon winding and allows the small wheel to fly away from the revolvir wheel and against a brake which immediately stops it. After th shock is received at all four hydrophones, therefore, the three dia indicate the time intervals between the arrival at the first hydro phone and that at the other three directly in thousandths of second, each thousandth representing a distance of about 5 ft, fro which the graphical diagram shown in fig 23 can be constructe and the position of the source indicated on a chart

In order to obtain this position as readily as possible the writ has devised what he calls a sound-ranging locator (fig 28, se plate facing p. 326). It consists of a long steel bar pivote at one end on a ball-bearing, the centre of which can be fixe on the chart exactly over the position of one of the hydrophone Three thin steel bands are attached to the other end of this bar b means of keys, like the strings of a violin, and pass through slot in a sliding piece to graduated rods sliding through simil ball-bearing swivels, which are fixed on the chart in positio corresponding to those of the other three hydrophones. Tl graduations on the sliding bars are marked in times to the sca of the chart, so that by sliding them to the readings correspondin to the time differences indicated on the chronometer, each stri is lengthened by the amounts bB, cC, and dD in the diagram fig. 2 and when the slotted slider on the main bar is pushed down an the bar turned until all the strips are tight, the point from whic they radiate indicates the position of the source on the chart witho any calculation, and a marking point just under the edge of th slider can be depressed to prick the position. In order to secu accuracy, each of the strips is provided with a small tension indi cator which shows when the strip is strained to a definite tensio

Two strips only are shown in fig. 28, but any number ma be employed according to the number of receivers.

A device on a similar principle has been put forward by Mr H. Dadourian in the United States.*

* *Physical Review*, August, 1919.

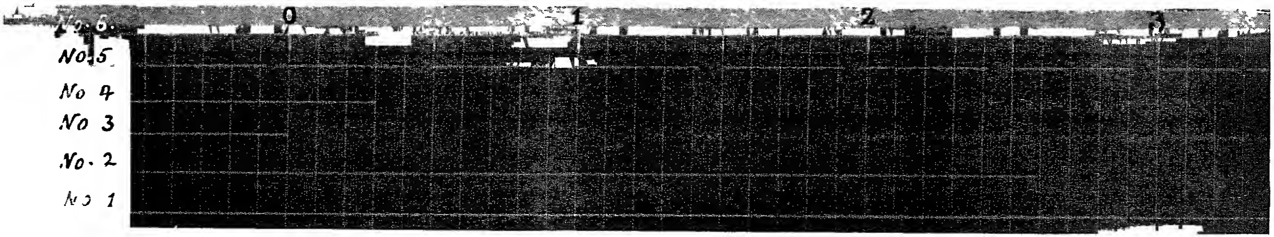
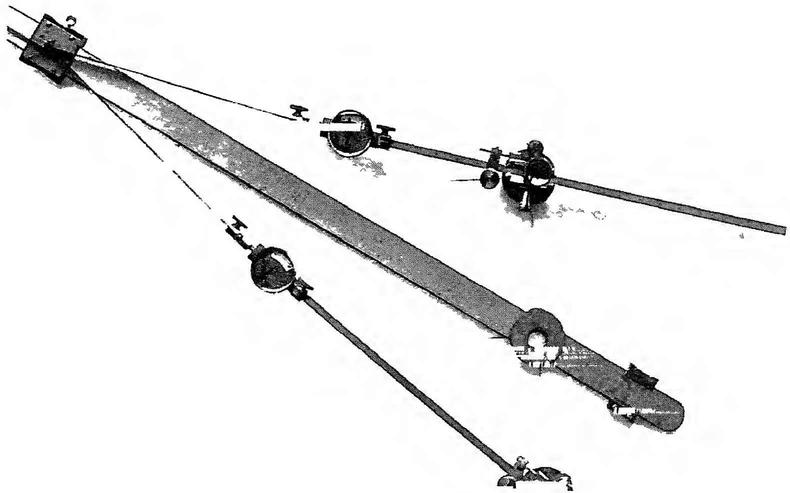


FIG 24 —SOUND-RANGING RECORD



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[n using the multiple-station method of sound ranging for ;ting navigation in foggy weather, a ship desirous of being rmed of its position calls up the nearest sound-ranging station, ;h instructs it to drop a depth charge As soon as the record eceived on the Einthoven camera or phonic chronometer, the tion of the ship is worked out or marked by the locator and lessed to the ship.

Wireless-Acoustic Sound Ranging.—A method of sound ;ing which promises to be of much greater value for navigation, which has not yet been fully developed as it was of little value in time, is the wireless-acoustic method proposed by Professor Joly. he original experiment of Collodon and Sturm in 1826, the city of sound in water was determined by striking an under- 'l bell and igniting a charge of gunpowder simultaneously knowing the distance from the source and observing the interval ime between the flash and the sound of the bell the velocity determined, as light travels practically instantaneously over any nary distance Conversely, if the time interval and the velocity known, the distance of the source can be at once determined, as he familiar method of ascertaining the distance of a lightning l by noting the time between the flash and the thunder clap The ntage of employing an underwater method is that sound is trans- ed more effectively through water, and that there are no water ents comparable with winds to affect the velocity appreciably Jnfortunately a flash of light is of no value in a fog, but wireless es are little affected by it, and travel with the same speed as , so that if a wireless flash and an underwater explosion are iated simultaneously at a lighthouse or other known position, the ship is provided with a wireless equipment and a directional ophone, the distance of the station can be at once determined he ship by noting the interval between the two impulses. As velocity of sound in sea water is nearly a mile a second, the ince can be determined within a quarter of a mile by a simple -watch, and the direction of the source found by either the ctional hydrophone or directional wireless, without any com- ication with the station. If the lighthouse or lightship simply s out wireless impulses simultaneously with the strokes of the narine bell at convenient intervals, all ships in the vicinity can e their positions from time to time without delay or mutual ference, and if they are within the range of two such stations can do so without any directional apparatus.

The recent developments in directional wireless have rendered the application of sound ranging to navigation of less importance, but even now wireless direction finding is not always reliable, especially at sunrise and sunset; and there is also liability to error on steel ships owing to their distorting effect on the wireless waves. As hydrophones become increasingly employed on ships for listening to submarine bells, &c , the ability to obtain accurate ranges by wireless acoustic signals will doubtless prove of great value

Leader Gear

Although not strictly speaking an acoustic device, some mention should be made of the leader gear or pilot cables as an aid to navigation of harbours and channels in foggy weather For this purpose it is necessary to be able to follow some well-defined track with a latitude of only a few yards, so that sound ranging is inadequate But if a submarine cable carrying alternating current of sonic frequency, say 500 ~, is laid along the desired track, and the ship is provided with search coils with amplifier and telephones, the alternating magnetic field produced by the cable induces alternating electromotive forces in the coils, and thus gives a sound in the telephones when the ship is sufficiently near the cable By using two inclined coils on the two sides of an iron or steel ship it is found that the sound is loudest when the telephones are connected to the coil which is nearer to the cable, so that the ship can be steered along it, and keep a fairly definite distance to one side of it, so that vessels passing in opposite directions will not collide This device, which was first put forward by Mr C A. Stephenson of Edinburgh in 1893, was revived during the war by Captain J Manson, and is now coming into use both in this country and in the United States. An 18-mile cable has been laid by the Admiralty from Portsmouth Harbour down Spithead and out to sea.

Acoustic Depth Sounding

Another purely acoustic device which promises to be of considerable value to navigation is that of depth sounding by acoustic echoes from the bottom. If a ship produces an explosion near the surface, the sound travels down to the bottom and is reflected back as an echo, and for each second of interval between the explosion and the echo the depth will be half the velocity of sound or 2500 ft., say 400 fathoms. Various experimenters, notably

I. Marti in France, Herr Behm in Germany, and Officers of the merican Navy, have devised apparatus whereby the time between ring a detonator or other small charge under the ship and the eception of its echo from the bottom can be recorded on a high-peed chronograph, and very accurate results have been obtained.

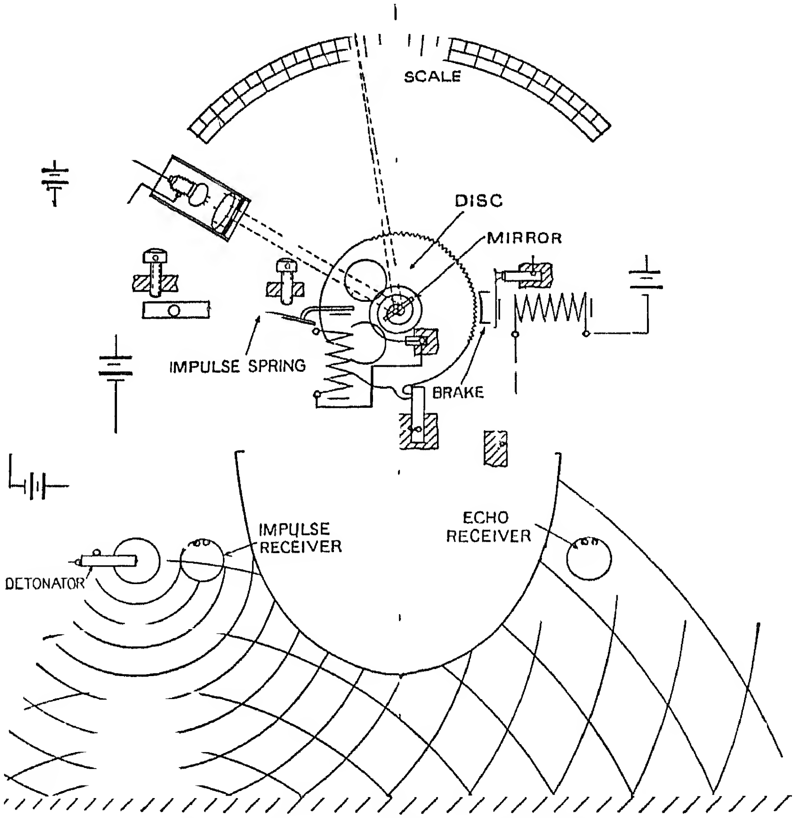


Fig 29 —Behm's Acoustic Depth-sounding Method

The method of Behm, called the "Echolot" or echo-sounding evice, now being developed by the Behm Echolot Co , Kiel, has ttained a high degree of perfection, and is claimed to give indiations in a ship at full speed, and even in rough weather, to an ccuracy of within a foot. The transmitter consists of a tube hrough which a cartridge is impelled by air pressure into a holder xed on the hull a little above the water line. The cartridge is red out of the holder on pressing the firing-key, and is shot towards

the Impulse Receiver, while a time fuse in the cartridge is arranged to explode a detonator just before the cartridge reaches the microphone. Both the Impulse and Echo Receivers are microphones, but the latter is screened from the direct effect of the detonator by being fitted on the opposite side of the ship.

The explosion of the detonator causes a sudden drop in the current through the impulse receiver and weakens the current passing round an electromagnet, and causes it to release an "impulse spring" which suddenly starts a pivoted disc in rotation with a uniform velocity until the weakening of the current through the brake magnet, due to the echo reaching the echo microphone, stops the disc. The angular motion of the disc is therefore proportional to the interval between pressing the firing-key and return of the echo, and a light mirror on the disc spindle causes a spot of light to revolve round a translucent scale divided in depths, and to stop at the depth indicated It is claimed that this timing device is capable of indicating short interval of time to an accuracy of one-ten-thousandth of a second, corresponding to only 3 in in depth Three keys are provided on the indicator, one for restoring the indicator to zero, one for firing the charge and obtaining the depth, and the third for checking the indicator against a standard time interval A number of detonator charges can be stored in the transmitter magazine, and fired as required The whole apparatus can be operated by a few dry cells, as the lamp is lit only at the moment of restoration, indication, or checking, and the colour of the light is varied at each operation to eliminate risk of mistake It is stated that a rock with an upper surface of only 2 sq. metres in area is sufficient to give a correct indication

The British Admiralty have recently developed a very simple and accurate echo sounding gear.

Echo Detection of Ships and Obstacles

By means of leader gear, sound ranging, and echo sounding navigation in fogs may be made much safer and more regular, but there still remains the great danger of collision in the open sea between ships, and especially with wrecks, rocks, and icebergs. As far as ships are concerned the difficulty is to some extent met already by signalling with sirens, but the curious blanketing and reflecting or refracting effect of fogs is a source of considerable confusion and danger. Underwater signalling does away with this difficulty almost entirely, and as hydrophone equipments become

ıore common the ıısk of collısion between moving ships will rapidly ıminısh.

With a good dırectional hydrophone equipment an ordınary eamship can easıly be detected and its direction determined up › a range of some mıles merely by the noise of ıts engines. But ı the case of wrecks, ıocks, and icebergs, whıch emıt no sound, ıe danger is stıll very great, and nothing but an echo method wıll etect them. Unfortunately thıs is a difficult matter, as a shıp or nall rock at a moderate dıstance is a very small target for an echo, › that the echo ıs of very small ıntensıty, and ıt may quıte easıly › masked by bottom echoes However, Fessenden, by the use of s powerful electromagnetıc transmitter, succeeded as early as)16 ın obtaınıng echoes from dıstant obstacles, and by employing rectional transmıttıng and ıeceıvıng devıces, whıch concentrate the und ın the desıred dırectıon, the strength of the echo can be creased, dısturbances reduced, and the dıectıon and approxımate nge of the obstacle determıned. As eaıly as 1912, just after the *tanıc* dısasteı, a pıoposal to employ echo detectıon for avoıdıng nılar dangers was put foıwaıd by Mr. Lewıs Rıchardson, and ıt ay be hoped that thıs method will ultımately elımınate the last the serıous dangers of navıgatıon

Acousıic Transmission of Power

Beloıe concludıng thıs article, reference ought to be made to the onderful achıevements of M Constantınesco, as showing the possıbılıtıes of what may be called acoustıc engıneering. For the purposes of underwater signallıng the power transmıtted, although large comparıson wıth what we have heretofore contemplated in conction wıth sound, raıely exceeds a hundred watts; and ıt has been t for M Constantınesco boldly to envisage the possıbılıty of ınsmittıng large amounts of power by alternating pressures in ıter of suffıcıently hıgh frequency to be descrıbed as sound waves. ır many yeaıs ıt has been customary to ıllustrate the phenomena alternatıng electrıc cuırents by hydraulıc analogıes, and the esent wrıter has even wrıtten a book in whıch such analogıes ve been used as a means of gıvıng a complete theory of the subject; t the obvıous possibılıty of usıng such alternatıng pıessures in ter for practıcal purposes was entırely missed untıl M. Conıntınesco conceıved ıt, and immediately the idea occurred it was ıdent that the whole of the theory was ready to hand from the

electrical analogies. In a surprisingly short time, therefore, M. Constantinesco has been able to devise generators, motors, and transformers capable of dealing with large amounts of power transmitted by hydraulic pipes in the form of acoustic waves of a frequency of about 50 ~. The generator is, of course, simply a high-pressure reciprocating valveless pump, and the motor can be of similar construction, but by having three pistons with cranks at 120°, three-phase acoustic power can be generated and employed in the motors. The first commercial application of M Constantinesco's devices has been to reciprocating rock drills and riveters, for which this method is especially suitable, as the reciprocating motion is obtained simply from a cylinder and pistol without any valves whatever, and the power is transmitted by a special form of flexible hydraulic hose pipe comparable with an electric cable It is not too much to say that M. Constantinesco's ideas have opened up an entirely new field of engineering, and their development may have far-reaching effects.

For a discussion of the theory of hydraulic wave transmission of power, see Chapter VI.

Although this article is necessarily very incomplete, it will at least have served its purpose of showing the great importance of underwater acoustics, and there can be no doubt that a new department of scientific engineering has been opened up which has vast possibilities

Developments in Echo Depth-sounding Gear.

Since the first appearance of this volume, the chief advance in underwater acoustic devices has been in the improvement of echo depth-sounding devices which have proved their great value for navigation and appear likely in time to become a standard feature of ship equipment Three different types of such gear are now manufactured in this country: the Admiralty type by Messrs H. Hughes & Sons; the Langevin piezo-electric type by the Marconi Sounding Device Company, and the Fathometer gear, which has been developed from the original Fessenden apparatus by the Submarine Signal Company. All these devices have now been made to give both a visual indication of the depth on a dial and a continuous record on a chart.

The basis of all methods of acoustic depth sounding is the recording of the time taken for a signal to travel from the ship to the bottom of the sea and return, but they differ in the type of the signal

nd method of indication, and may be divided into impulse or sonic " methods and high frequency or " supersonic " methods n the former class to which the Behm " Echolot " (see p. 331), e original Admiralty sonic gear, and the Fathometer belong, the gnal is in the form of a single powerful impulse provided by an xplosive cartridge or an electromagnetic or pneumatic hammer riking a diaphragm; while in the latter a short train of high-equency vibrations is emitted from a quartz piezo-electric oscillator, steel rod which vibrates at a high frequency when struck by a ammer, or by a magnetostriction oscillator which is the magnetic nalogue of the quartz oscillator

The single impulse or sonic transmitter is practically non-irectional, i.e the disturbance travels equally in all directions under e ship. This has the advantage of making the indications practi-illy independent of any rolling of the ship, but it has many dis-dvantages Firstly, it is liable to give such a severe shock to the ceiver at the moment the impulse is sent out that it does not cover in time to respond to an echo from a very shallow bottom, condly, the greater part of the energy is wasted, thirdly, the echo ust be very strong to be heard above the noises caused by the ship's achinery and motion through the water, and fourthly, it may not ive true depths if the bottom is shelving steeply, as the first echo received from the object which is nearest to the ship With the igh-frequency method the sound can be concentrated within a one of any desired angle, so that the receiver can be fairly close to e transmitter without sustaining any severe initial shock, and the ceiver can be sharply tuned to the transmitted frequency, so that is nearly deaf to any other disturbances If the ship could be kept n a perfectly even keel, the narrower the beam the better, as it ould be equivalent to a vertical sounding line, but on account of olling it is desirable that it should have an angle of something like alf the maximum angle of roll. For a circular transmitter the miangle of the beam θ is given by the relation $\sin\theta = 1{\cdot}2\frac{\lambda}{d}$ where λ the wave length of the sound and d the diameter of the trans-itter, so that we can obtain any beam angle we please by varying e diameter and frequency

As regards receivers, a granular microphone is the most suitable r the single impulse or sonic system, and it must be mounted at me distance from the transmitter and preferably on the other side the keel, so as to be shielded as much as possible from the initial

shock. This separation is however objectionable, as it seriously reduces the accuracy of sounding in very shallow water, where it is frequently most important. The device, of course, indicates the distance from the transmitter to the bottom and back to the receiver, and this varies very little when the depth under the keel is small compared with their separation. With the high-frequency system, however, the transmitter and receiver can be close together, so that this difficulty does not arise; and as both the quartz and magnetostriction transmitters will also serve as receivers, it is even possible to dispense with a separate receiver, as is done in the Marconi gear

The essential function of the indicator is, of course, the measurement of the time interval between the impulse and echo As the average velocity of propagation of sound in sea water is about 4900 ft. per second, and the sound has to travel the double distance to the bottom and back, each second of interval corresponds to a depth of 2450 ft. or about 400 fathoms; and if soundings are required within an accuracy of one foot, the time must be measured within an accuracy of four ten thousandths of a second The most simple and reliable method of effecting this is by the contact method employed in the Admiralty sonic gear, in which the receiving earphones are shunted by two brushes, which press on a revolving ring which has a small gap in it, so that the phones are short-circuited for all but an interval of one or two thousandths of a second in each revolution The transmitter is actuated at a certain moment in each revolution, and the two brushes are carried on an arm which can be turned by the observer until the short circuit is removed simultaneously with the arrival of the echo. The depth is then indicated by the position of the arm on a scale which can be divided in feet or fathoms Direct visual indication is, of course, preferable, and is secured in the Fathometer gear by a revolving disc mounted close behind a ground-glass scale The disc has a narrow slot in it, behind which is a small neon lamp, and the echo when sufficiently amplified, causes this lamp to flash and show a momentary red streak on the scale at each revolution. In the Marconi gear the amplified echo is received by an oscilloscope or high-frequency galvanometer the beam from which falls on an oscillating mirror and shows a luminous streak on a ground-glass scale. When the echo is received the momentary kick of the galvanometer shows as a kink in this luminous streak at the corresponding depth on the scale. Messrs. Hughes have produced a direct-reading pointer indicator for the high frequency Admiralty gear, which operates on the phase indicator principle. A revolving

solenoid is fed with direct current and therefore produces a rotating magnetic field, and a soft iron needle is momentarily magnetised by the current from the echo receiver, so that it sets itself along the axis of the solenoid at that moment.

Any of these devices enable the depth to be observed at intervals of every few seconds even when the ship is running at full speed, which is an enormous advantage over the old lead line, which required the ship to be running dead slow Merely for ensuring safety in navigating shallow waters this is sufficient, but a great gain is secured by making the apparatus record the depths continuously on a chart which gives a profile of the bottom along its course, as this enables a ship to locate its position with considerable precision if the con-

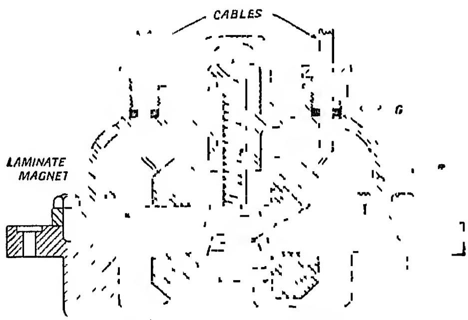


Fig 30 —Electromagnetic Hammer Transmitter for Shallow Water Gear

tour of the bottom is accurately known. During the last few years recorders have come into general use, and have been found very satisfactory The motor which actuates the transmitter contacts and the receiver mechanism is also employed to move a stylus uniformly across a band of paper which has been previously soaked in a sensitive solution (usually starch and potassium iodide, as in the early Bain printing telegraph), and the amplified and rectified echo causes it to make a mark on the paper at the moment it is received The paper band is moved slowly forward at a constant rate by the same motor, or it can be driven from an electrical log so as to move proportionately to the distance covered by the ship, and, as the stylus makes a mark for each echo, a practically continuous line is drawn on the paper showing the variation of depth either with time or distance. By simple contact devices the stylus can also be made to mark the paper at each five or ten feet or fathoms of depth, and at regular intervals of time or distance, so that the record is complete,

and can be reproduced directly in a hydrographic atlas. Fig. 31 shows such a record of a 15 minutes' run, with a shallow water magnetostriction set.

After the above general description, the only features of the various gears which require special consideration are the trans-

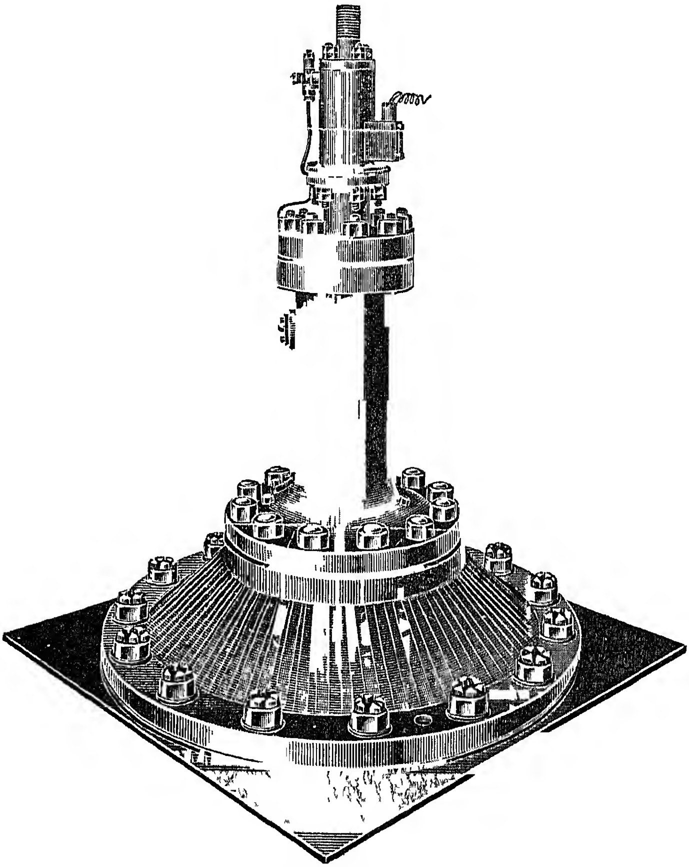
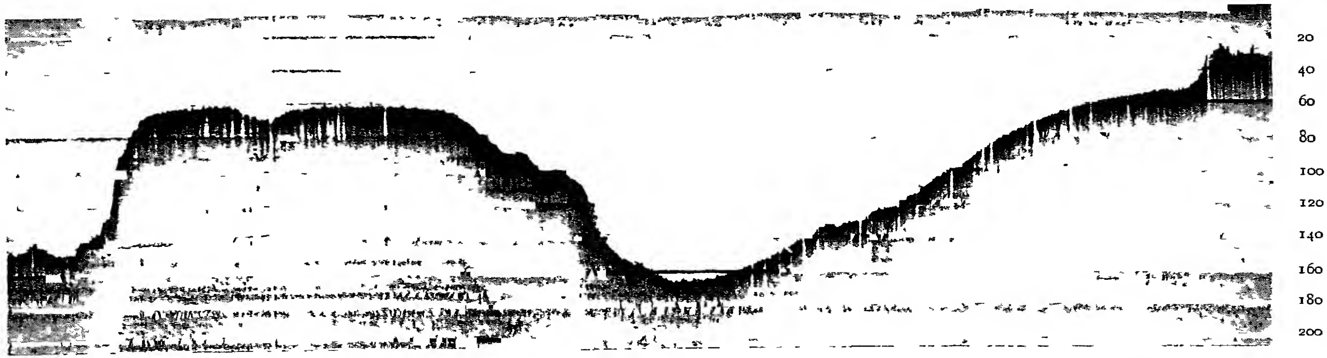


Fig 32.—Pneumatic Hammer Transmitter

mitters. For the impulse or sonic transmitters the types employed in the shallow water Admiralty gear and the Fathometer gear are very similar, and the former is shown in fig. 30. A ring of iron stampings, with internally projecting poles, is excited by coils on the poles, and the hammer consists of a tapered block of stampings, which is drawn into the gap between the poles and compresses a spiral spring which drives the hammer down against a diaphragm when the



H Hughes & Son Ltd

FIG 31 —DEPTH RECORD

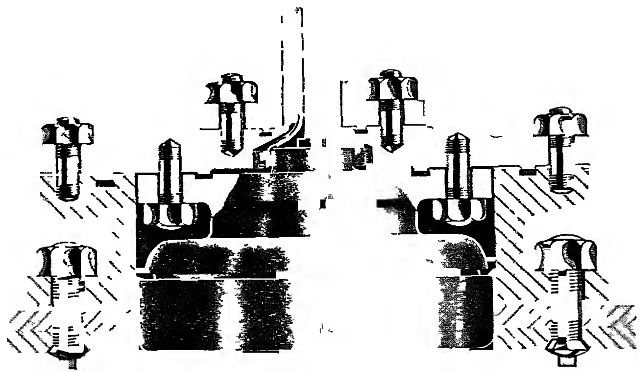


FIG 33 —MARCONI QUARTZ TRANSMITTER

rent is broken. For the deep water Admiralty gear, which has ɜn used for depths of over 2000 fathoms, the hammer is operated ɜumatically with an electromagnetic release (fig. 32).

The Marconi high-frequency quartz transmitter, which also ves as the receiver, shown in figs. 33 (p. 338) and 34, has a thin er of quartz crystals H cemented between two steel discs F and G,

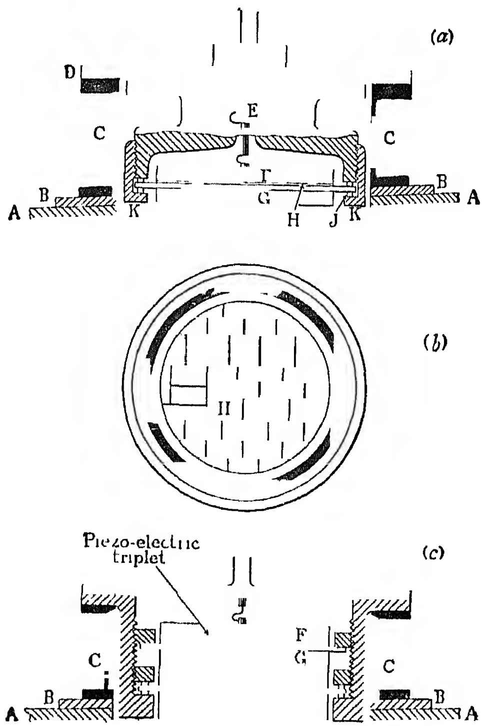


Fig 34 —Marconi Quartz Transmitter

e lower of which is usually in contact with the water while the upper highly insulated and connected to a high-voltage oscillator which ves a frequency of about 37,500 periods per second, producing ives about 4 cm long in the water, and a somewhat sharp beam ι order to provide for the removal and replacement of the oscil- tor without dry-docking the ship, the housing is sometimes pro- ded with a second resonant steel plate shown at the bottom of C hich is clamped by a central flange and transmits the oscillations the water.*

* Cdr. J. A. Slee, C.B.E., R.N., *Journal Institution Electrical Engineers*, Dec. 1931.

Quartz oscillators, although highly efficient, are somewhat costly and require special technique in construction, and hence efforts have been made to obtain a high-frequency impulse without employing crystals. One simple method, which is fairly effective, is to employ the ordinary hammer of the single-impulse transmitter, but to substitute a steel rod clamped at its centre like the lower plate in the Marconi transmitter for the diaphragm. This rod

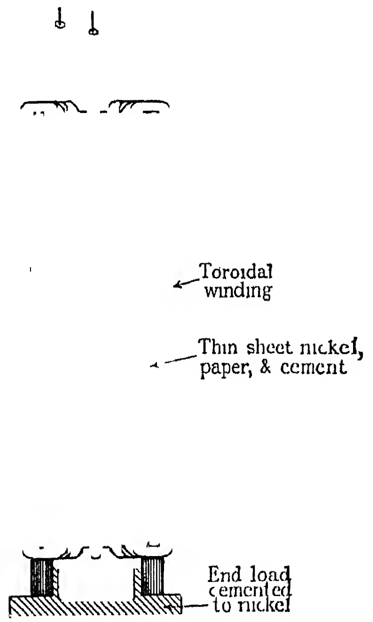


Fig 35 —Magnetostriction Scroll-type Oscillator

vibrates with its resonance frequency and emits a short train of damped oscillations each time it is struck

But within the last few years a great advance has been made by employing the principle of magnetostriction, i.e. the deformation which takes place in magnetic materials when they are magnetized. This effect is most marked in pure nickel, and in certain nickel and cobalt steel alloys, and it enables oscillators of any frequency to be constructed very cheaply and by ordinary workshop methods It lends itself to very various forms of oscillator, but the two which have been found most convenient for echo sounding work are the "scroll" and "ring" types shown in figs. 35 and 36. In the former, a strip of nickel is simply wound up like a scroll of paper, and is

rovided with a simple toroidal winding like a gramme ring armature. /hen this winding is supplied with alternating current the scroll xpands and contracts axially, so that a disc cemented to one end :rves as the emitting surface. The axial length of the scroll is made ich that its mechanical resonance frequency is that required

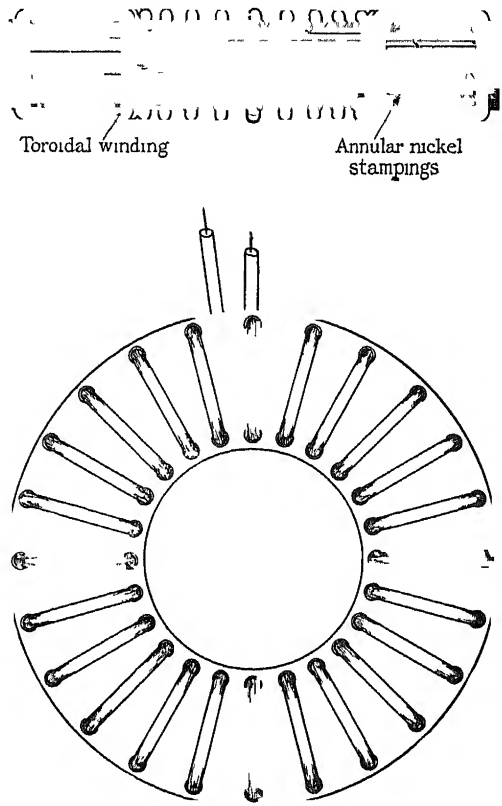


Fig 36 —Magnetostriction Ring Oscillator

isually about 15,000 periods per second), and the winding is fed with ternating current at a low voltage either from a valve oscillator or condenser discharging through an inductance, in either case at the sonant frequency In the disc type the oscillator is made up of a imber of ring stampings with holes round its inner and outer eriphery These stampings are cemented together into a solid ock with an insulating cement, and a toroidal winding is wound irough the holes. When supplied with alternating current the

ring expands and contracts radially, and it is operated at its resonance frequency. As the sound is emitted in all directions in a plane parallel to the ends of the ring, it is surrounded by a sound reflector made of two thin metal cones with india-rubber "mousse" between

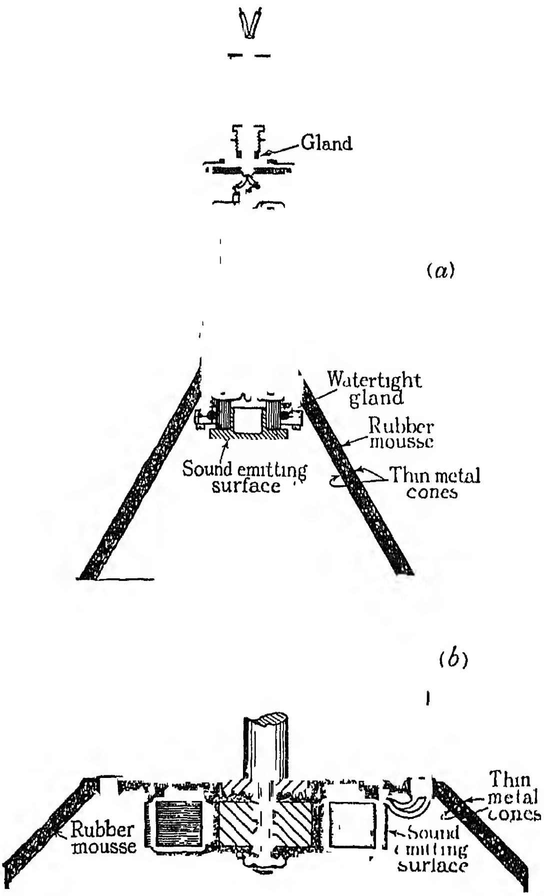


Fig 37 —Magnetostriction Oscillators and Reflectors

them, as shown in fig 37, and the beam angle can be varied by choosing the diameter of this reflector The transmitters will serve equally well as receivers, but it is found preferable to mount two of them close together, one acting as transmitter and the other as receiver

These magnetostriction transmitters and the associated recorder were designed by Drs. A. B Wood and F. D. Smith and Mr. J. A

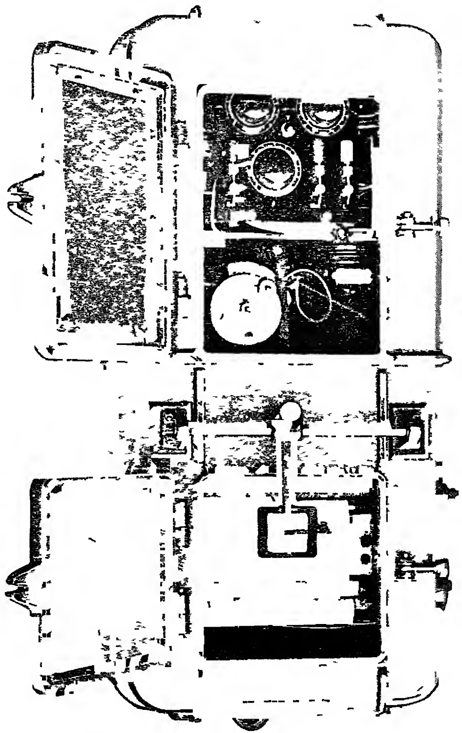


Fig 39 —Marconi Indicator and Recorder

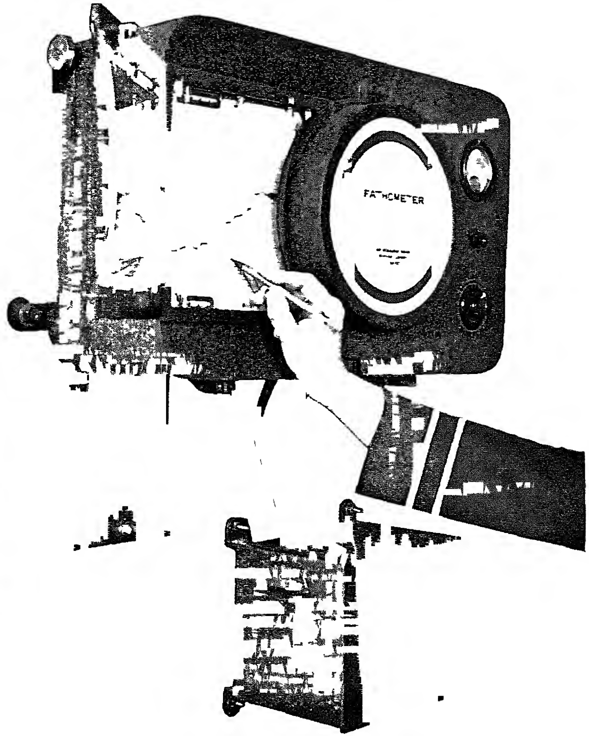


Fig 40 —Fathometer Indicator and Recorder

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Geachy,* of the Admiralty Research Laboratory, after a research magnetostriction by Dr. E P Harrison, and have been incor- ited with great success in the latest forms of Admiralty depth-

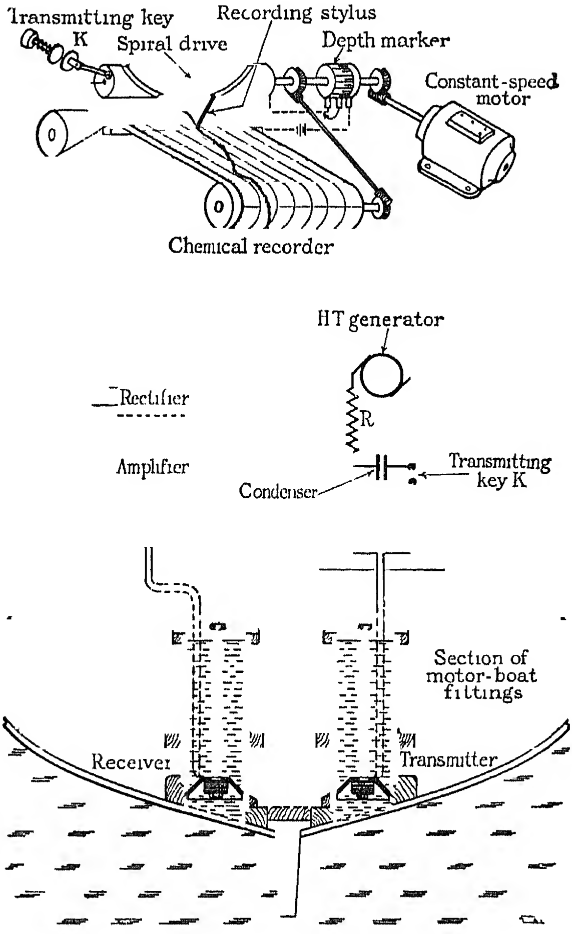


Fig 38 —Magnetostriction General Arrangement

inding gear. As they only require very small power for shallow)ths without high-voltage oscillators, it has been possible to instal m with recorders in small motor-boats for the hydrographic survey shallow rivers and estuaries, which has enormously increased the ility and rapidity of such surveys On the other hand, they have

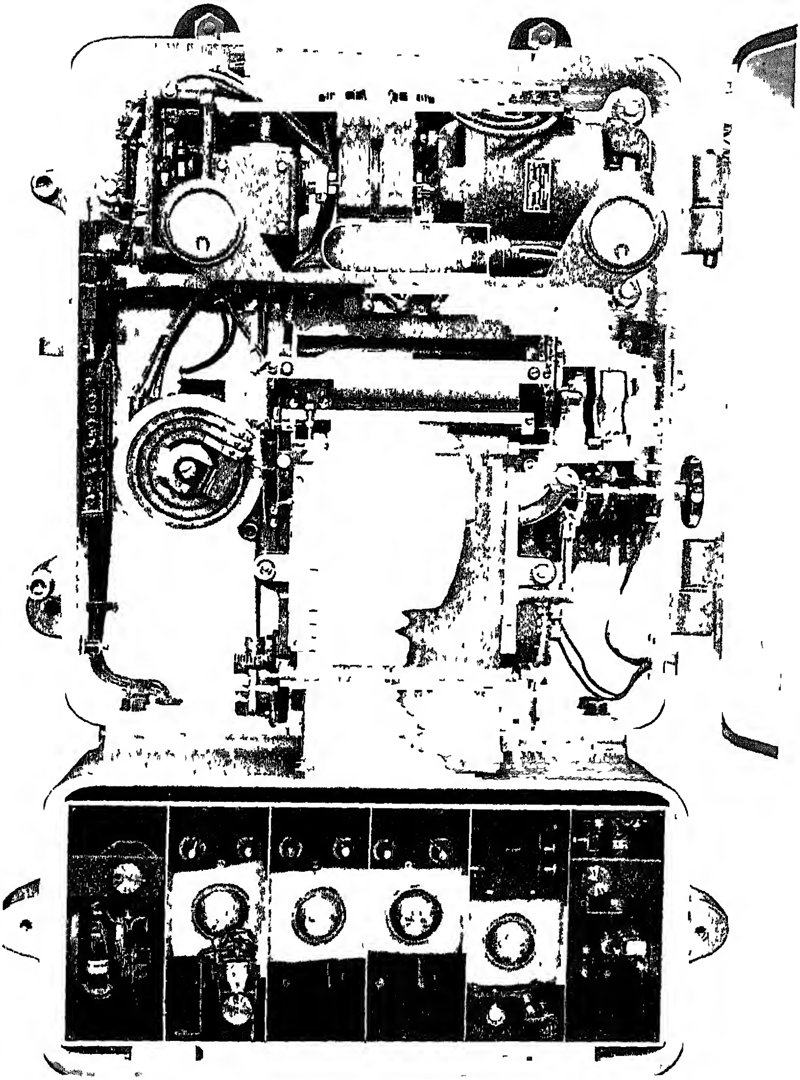
* *Journal Institution of Electrical Engineers*, Vol 76, No 461, May, 1935, p 550

proved equally efficient for moderate depths and deep water gear, and soundings have been taken successfully in depths of 2000 fathoms with the transmitters and receivers in water-filled tanks in contact with the hull, and transmitting and receiving through the ship's plates. Fig 38 is a diagram of the motor-boat outfit with chemical recorder

Figs. 39, 40 and 41 show external views of the Marconi, Fathometer, and Admiralty indicating and recording sets (Plates facing pp. 342, 346).

Probably over two thousand naval and mercantile vessels have by now been equipped with echo depth-sounding gear of one or other of the above types, and the reports on them show their great value, accuracy and reliability For moderate depths down to 200 or 300 fathoms, an accuracy of a foot is obtainable, while the magnetostriction motor-boat gear actually records depths to an accuracy of three inches even when the boat is almost touching bottom The deep water set on the *Discovery II* was so satisfactory in Antarctic waters that line sounding was discontinued, and such sets have been of great service in cable-laying ships by enabling them to lay their cables on the least irregular bottoms The importance of continuous and recorded soundings for mercantile ships by facilitating their entry to harbours and locating their positions at sea has already been referred to, and its value will increase as more and more records along the main trade routes become available Lastly, a remarkable application of such gear has been found for fishing, and many trawlers are now being equipped with it, as it has been found that the quantity and quality of catches depends greatly on the depth, and echo sounding gear enables the ship to follow a contour line of the depth desired. Echo depth-sounding has proved the most useful application of underwater acoustics and seems likely to be universally adopted

The writer is indebted to the three firms above mentioned for particulars and illustrations in this section, and to the *Journal of the Institution of Electrical Engineers* for figs. 34, 35, 36, 37, and 38.



H Hughes & Son, Ltd

FIG 41 —ADMIRALTY SUPERSONIC RECORDER

Facing page 344

CHAPTER X

The Reaction of the Air to Artillery Projectiles

Introduction

All calculations of the motion of a projectile through the air are ·cted to one object—to determine the position and velocity of the jectile at any given time after projection in any prescribed manner. general the reaction of the air to a rotating projectile is very com- ated, the complication is considerably reduced, however, if the jectile can be made to travel with its axis of symmetry coincident 1 the direction of the motion of its centre of gravity. It is a ter of experience that by giving the projectile a suitable spin it be made to travel approximately in this manner for considerable ances; in such circumstances the reaction of the air is reduced to ngle force, called the *drag*, which acts along the axis of the pro- ile and tends to retard its motion.* When this drag is known for rojectile of given size and shape the problem enunciated above omes one of particle dynamics, and its solution for that pro- ile can be effected, at all events, numerically. The first and major of this chapter is devoted to the consideration of this drag.

When the angle of elevation of the gun is considerable the cur- ure of the trajectory increases too rapidly for these simple condi- s to hold. The motion then becomes complicated and the prob- becomes one of rigid dynamics in three dimensions; the trajectory twisted curve instead of a plane one, and the well-known pheno- ion of *drift* appears Similar complications arise when the pro- ile is not projected with its axis coincident with the direction of ion, or when the spin is insufficient to maintain this coincidence

A couple of small magnitude due to skin friction also exists, it acts about the and tends to reduce the spin; its effect is generally negligible with modern ·ctiles.

In the second part of this chapter the component forces and couples of the reaction of the air in these circumstances are briefly considered.

THE DRAG

Early Experiments: the Ballistic Pendulum

Most early writers on ballistics* assumed that the resistance of the air (the drag) to the motion of projectiles was inconsiderable The first experimenter to attempt the determination of the air drag on projectiles moving at a considerable speed was Robins, who, in 1742, carried out experiments with his ballistic pendulum He found that the resistance encountered was abnormally greater for velocities greater than about 1100 ft. per second than for lesser velocities Following Robins, many experiments were performed with the ballistic pendulum, notably at Woolwich (by Hutton, 1775–88) and Metz (by Didion, 1839–40), to determine the drag as a function of the velocity of the projectile †

The method employed by Robins was, briefly, as follows:

A gun was placed at a known distance from a heavy ballistic pendulum; the charge was carefully weighed and the projectile was fired horizontally at the pendulum. The latter received the projectile in a suitable block of wood, and the angle through which it swung was recorded Knowing the weights of the projectile and pendulum and the free period of oscillation of the latter, the velocity of the projectile at the moment of hitting could be calculated The experiment was repeated with the same charge, the distance between the pendulum and gun being varied from round to round. There resulted a series of values of velocity at known distances from the gun, the retardation of the projectile and hence the resistance of the air at these distances could be deduced

By performing similar sets of experiments with various weights of charge, the drag could be determined as a function of the velocity of the projectile.

The uncertainty of realizing the same muzzle velocity in each set of experiments with constant charge vitiated the reliability of the results. Hutton overcame this difficulty by hanging the gun hori-

* The study of the flight of projectiles

† For a full account of these experiments see Robins, *New Principles of Gunnery*, 1761; Hutton, *Tracts*, 1812, especially Tract XXXIV; Didion, *Lois de la résistance de l'air* (Paris, 1857).

ıtally from a suıtable support, so that the gun itself became the ɔ of another pendulum From the angle through which thıs tem swung on firing, the muzzle velocity of the projectile could calculated. For each round fired he thus obtaıned two values the velocity—one at the muzzle, the other at a known distance m the muzzle

Let v_1 and v_2 be these values, and let x be the distance between ռ and pendulum. Then, if m be the mass of the projectıle, the s of energy ın traversıng the distance x ıs $\frac{m}{2}\left(v_1^2 - v_2^2\right)$. If R the mean value of the drag we therefore have

$$R = \frac{m}{2x}\left(v_1^2 - v_2^2\right).$$

ɔvided that the dıstance x ıs sufficiently small, this value may be ..en as the actual value of the drag for the velocıty $v = \frac{1}{2}(v_1 + v_2)$

By varyıng the charge and the dıstance between the gun and ndulum Hutton determıned the drag numerıcally as a function of ɜ velocıty

From the tıme of Hutton to the present day, experıments concted on the Contınent and ın Amcııca to measure the ıesıstance the aır have been based on thıs pııncıple, namely, to determıne ꞉ velocıty at two poınts on an approxımately horızontal trajectory known dıstance apart A large number of ınstruments for mearıng the velocıty of a projectıle at a gıven poınt have been ınvented ıırıng this tıme; the reader is refeıred to *Balístıca Experımental Aplıcada*, by Col Negrotto of the Spanısh aımy (Madrıd, 1920), r an up-to-date and exhaustıve account of them. It should be ıted here that few of these chronogıaphs were invented especıally r the determınation of the resıstance of the aır; there are, of course, any ımportant uses for such ınstruments ın gunnery.

Sınce 1865 experıments on the resistance of the air conducted England have been based on a dıfferent principle. The method ıs first proposed by the Rev. F. Bashforth, B.D., sometime Proɜssor of Mathematıcs at the Artillery College, Woolwıch; ıt consısts measuring the times at which a projectıle passes a number of uıdıstant points along an approximately horizontal trajectory. hese tımes are then smoothed and dıfferenced, and the velocity and tardatıon of the projectile at a number of correspondıng points are lculated by the method of finite dıfferences. This method is

evidently more economical in expenditure of ammunition than that of foreign experimenters.

The Bashforth Chronograph

In 1865 Bashforth invented his now-famous electric chronograph,* by means of which he succeeded in measuring small intervals of time with an accuracy previously unattained in ballistic instruments.

The chronograph consists essentially of two electro-magnets, to the keepers of which two scribers are attached by linkwork; these scribers trace continuous spiral lines on paper fixed on a revolving cylinder. The two spirals are generated by a mechanical movement of the framework supporting the electro-magnets in a direction parallel to the axis of the cylinder. The movement of each keeper is controlled by a suitable spring, and any small movement of either is identified on the record by a kink on the corresponding spiral trace.

One of the electro-magnets is connected with a clock and the current is broken momentarily every second; one of the spiral traces thus constitutes a time record. The other electro-magnet is connected in series with screens placed at equal distances along the trajectory of the projectile. At the moment the latter passes a screen the current is broken for a short interval of time and a kink is made in the corresponding spiral trace. The mechanism by which the current is broken is as follows.

A board is supported in a horizontal position with its length at right angles to the direction of motion of the projectile. Transverse grooves are cut in the board at equal distances, somewhat less than the diameter of the projectile. Hard spring-wire staples are fixed in the board so that each prong projects upwards from a groove.

On the near edge of the board a number of copper straps are fixed; each strap has two oval-shaped holes which are placed at the near ends of adjacent grooves. The prongs of the staples are bent down into the grooves and project through the oval-shaped holes; the arrangement is such that the butts of the staples and the copper straps alternate, so that a current may pass continuously through the staples and straps.

The prongs terminate in hooks from which are suspended small

* For a full account of Bashforth's experiments consult his *Revised Account of the Experiments made with the Bashforth Chronograph* (Cambridge University Press, 1890).

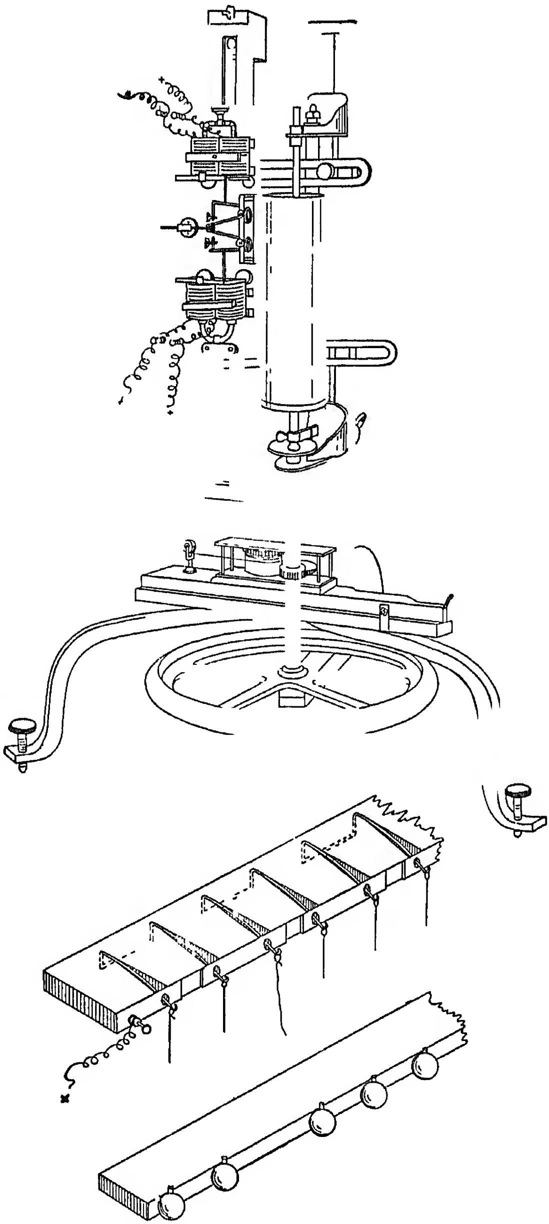


FIG 1 —The Bashforth Chronograph and Screen Reproduced by courtesy from "Description of a Chronograph", by F Bashforth, B D, *Proceedings of the Royal Artillery Institution*, 5, 1867

equal weights by means of fine cotton; the weights rest against a second horizontal board supported some distance below the first, and are sufficiently heavy to maintain the prongs in contact with the bottom edges of the holes in the straps.

When the projectile passes it will break at least one of the cottons; the corresponding prong will spring from the bottom to the top of its hole in the copper strap and so break the current momentarily. This mechanism constitutes a " screen ".

The record, when removed from the cylinder and laid flat, consists of two parallel straight lines with kinks in them; in the upper line the kinks correspond to the passage of the projectile through successive screens; in the lower the kinks indicate seconds of time. With a suitable measuring apparatus it is possible to read off the time intervals between the screens to four decimal places of a second.

Bashforth continued his experiments until 1880, and produced a table giving values of the air drag for velocities up to 2780 ft per second. Contemporary experiments were also conducted in Europe by Mayewski (Russia), Krupp (Germany), and Hojel (Holland), giving results in substantial agreement with those of Bashforth

Later Experiments

In the early years of the present century a large amount of work was done in England, France, and Germany to obtain accurate information concerning the air drag In 1906 the Ordnance Board used a method similar to that of Bashforth, but having a more accurate timing and recording device; the drag for velocities up to 4000 ft per second was determined. In 1912 O von Eberhard, at Krupp's,* made a large number of experiments with projectiles of various shapes and sizes. It is thought that these form the most exhaustive set of experiments yet undertaken; the results, which are frequently used in this chapter, are certainly the most complete yet published openly.

In this method the velocity at two points on the trajectory was measured by means of a spark chronograph; the distance between the points varied from 50 m. for small projectiles to 3 Km. for those of large calibre The method of deducing the resistance was similar to that used by Hutton, and results for velocities up to 1300 m. per second were obtained.

* Cf O von Eberhard, *Artilleristische Monatshefte*, 69 (Berlin, 1912).

Krupp's 1912 Experiments

The velocity at a given point is measured, in this method, by iring the projectile through two screens; one screen is placed a neasured distance (a few metres) in front of the point, and the other he same distance behind it The spark chronograph measures the ime taken by the projectile to traverse the distance between the creens; the average velocity between them is deduced and is taken ɔ be the actual velocity at the point. The distance being short o appreciable error arises from this assumption.

Each screen consists of a square wooden frame across which fine opper wire is stretched backwards and forwards continuously in

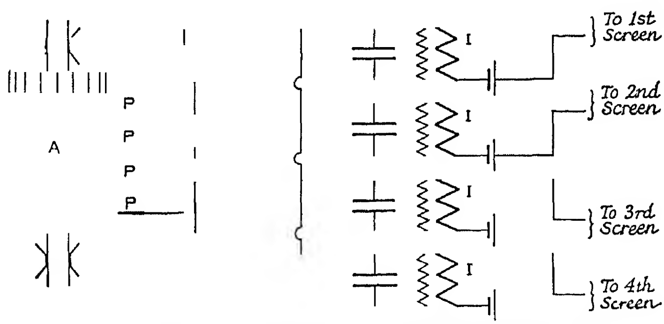


Fig 2 —The Spark Chronograph used in Krupp's 1912 Experiments

ich a way that a projectile passing through is certain to break the 'ire. In the experiments four screens are used; one pair serves › measure the velocity at the beginning of the measured range, the ther, the velocity at the end of it.

The spark chronograph is shown diagrammatically in fig 2 A ietal drum A is rotated at high speed by means of a suitable motor, ie speed being recorded by means of a Frahm tachymeter; readings › within one revolution per second can be taken with this instrument. It is essential that the speed of the drum be constant during ie flight of the projectile over the range, and this instrument serves ie additional purpose of indicating the most suitable moment to e the gun. The surface of the drum is silvered and is coated with ›ot except at one edge where a circumferential scale is fixed.

There are four induction coils, I; their primary circuits, which ›ntain batteries, are connected respectively to the four screens; ie terminal of each secondary circuit is connected to the spindle of ie drum, the other terminals being connected to sharp platinum

points, P. A break in one of the primary circuits will cause a spa to pass across the small gap between the corresponding point and tl drum; the spark is enhanced by a condenser in parallel with tl secondary circuit. The mark on the drum made by the spark like a bright pin-point and is surrounded by a sort of halo; it is thı easily identified.

The positions of the marks are read by means of a microscop mounted on the frame supporting the drum; this microscop can be traversed parallel to the axis of the drum. To take a readin the drum is rotated by hand until the mark made by a spark is i the field of the microscope; it is then clamped. The mark is the brought to the zero line of the eyepiece by means of a fine adjustmen The microscope is then traversed to the edge of the drum and tl reading is taken from the circumferential scale The positions (the marks made by the other sparks are similarly measured Tl time intervals between the pairs of sparks are then deduced wit the aid of the tachymeter reading

The chronograph is calibrated by breaking all the primar circuits simultaneously and recording the relative positions of tl marks made by the sparks.

With this instrument such a small time interval as 0·0017 se can be measured with a probable error of $7\ 5 \times 10^{-6}$ sec., or 0 4 per cent.

Experiments prior to the war thus fell into two types—the Hutto type, in which the velocity of a projectile was measured at two point a known distance apart, and the Bashforth type, in which the times c passing a number of equidistant points along the trajectory wer recorded.

With regard to the first type, unavoidable errors in the measure ment of the velocity would vitiate the results if the distance betwee the points were too short; on the other hand the approximate metho of deducing the resistance as a function of the velocity cannot giv satisfactory results unless the distance is short. It would thus appea to be a difficult matter to choose suitable distances, and laws of re sistance based on methods of this type must be somewhat uncertair

With regard to the second type, it is evident that, provided th time readings are sufficiently smooth to ensure that differences c finite order vanish, the resistance and velocity at correspondin points can be deduced to a known degree of accuracy. Wher however, the observed times require appreciable alteration to mak

smooth, considerable uncertainty attaches to the results deduced. 'o both types there is the objection that no account is taken of ble yaw * of the projectile. It is well known that all shells have yaw on leaving the muzzle of the gun, and it cannot be hoped it is always damped out sufficiently before reaching the points ich observations are made. At high velocities very considerable nay develop, and, in particular, such obstacles as screens may to increase it. In any case the yaw does not remain constant g the flight of the projectile, hence, unless it is at all times gible, the resulting law of resistance cannot be consistent.

i any method which depends on observations of a projectile in it is therefore necessary to make some provision for observing iw as well as the velocity (or time) at points on the trajectory. e yaw is small throughout the flight reliable results would be ied; in cases of considerable yaw a method would have to be ed of correcting for it in the analysis of the records before any ce could be placed on the deduced values of the resistance

Cranz's Ballistic Kinematograph

method in which provision is made for observing the yaw, at qualitatively, was devised by C Cranz,† who carried out experiments which were contemporary with those of Eberhard, it was, ver, applicable only to rifle bullets and to similar projectiles of small calibre

series of shadow photographs of the bullet was taken by means Ballistic Kinematograph at each end of a 20-m range. The ty at each end could be deduced from the positions of the ssive images on the kinematograph film and the observed speed ich the film moved through the camera The occurrence of ciable yaw could be detected at once from the photographs, ie reliability of the records for the purpose of the experiments thus be estimated.

The Solenoid Method

ice the War technique has developed considerably in the rement of high velocity. A very successful method, developed Frank Smith, of measuring time intervals in experiments of

he yaw is the angle between the axis of the shell and the direction of motion entre of gravity

)r a full account of Cranz's experiments see *Artilleristische Monatshefte*, **69** 1912).

the Bashforth type, consists in firing an axially-magnetized projectile through the centres of a series of equidistant solenoids which are connected in a series with a sensitive galvanometer. The current induced in each solenoid reaches a maximum as the projectile approaches, falls rapidly to a minimum as the projectile passes through, and finally returns to its original value as the projectile emerges. The " signature " of the galvanometer is recorded photographically on a rapidly moving film on which is also recorded the oscillations of a tuning-fork of known frequency.

Experiments with High-velocity Air Stream

During the war experiments were undertaken in a new direction. Instead of making observations on a projectile in flight, the thrust on a stationary projectile in a current of air moving at high velocity was directly measured.* The method has subsequently undergone considerable development in France and America,† and at the National Physical Laboratory ‡

The projectile (or a scale model) is supported by means of a thin steel spindle fixed to the centre of the base in prolongation of its axis; this spindle is attached at its other end to a mechanism designed to measure the thrust on the projectile. Compressed air issues from a reservoir through a suitable orifice, thus generating a high-velocity stream; the projectile is placed in the centre of the stream. The determination of the most suitable size and shape of the orifice was a matter of considerable difficulty, after a large number of trials an orifice was obtained which ensured a steady stream in the vicinity of the projectile.

The temperature and velocity of the air in the stream are computed on the classical theory from the state of the air in the reservoir before the orifice is opened. When the velocity is greater than that of sound in air a check on the computed value can be obtained by photographing the head wave caused by the projectile (see p. 357) and measuring its slope.

The possibilities of such an experimental method are innumerable. Apart from the direct determination of resistance of air at all angles of yaw, its sphere of usefulness extends to the elucidation of many problems connected with the general reaction of the air on projectiles.

* Experiments of various kinds on projectiles had previously been carried out in wind channels, but the velocity of the air stream was at most 30 m. per second.

† See " The Experimental Determination of the Forces on a Projectile ", by G. F Hull, *Army Ordnance*, Washington, May–June, 1921.

‡ See the Annual Reports of the Director, N.P.L. for 1922 and succeeding years.

Considerable progress has been made in experimental ballistics nce the war. The time, the yaw, and the orientation of the axis of ie projectile at a series of points along a horizontal trajectory can e observed with considerable accuracy; results from such experiients co-ordinated with those of experiments with the high-velocity r stream, have placed our knowledge of the reaction of the air on projectile upon a very sound foundation.

The Drag at Zero Yaw

The resistance of the air to a projectile of given shape, moving ith its axis of figure coincident with its direction of motion, depends ı the following arguments:

the velocity, v, of the projectile;
the calibre (i e. diameter), d,
the characteristic properties of the air, chiefly the density, the the elasticity, and the viscosity

Dimensional considerations lead us to the form

$$R = \rho d^2 v^2 f(v/a, vd/\nu) \quad . . \quad . \quad . . . . \quad (1)$$

r the resistance R of the air to projectiles of given shape, where ρ the density of the air, a is the velocity of sound in air, an index the elasticity, and ν is the kinematic viscosity The function $v/a, vd/\nu)$ in this expression is called the *drag coefficient* Since $^2v^2$ has the dimensions of a force it follows from equation (1) that e drag coefficient has no physical dimensions, its arguments must erefore be so chosen that they shall have no dimensions; v/a and $/\nu$ both satisfy this condition, and are the simplest arguments in ms of which the function can be expressed.

The Drag at Low Velocities

For velocities below the critical velocity it is well known that

$$f(v/a, vd/\nu) = A\nu/vd$$

ere A is a constant, the terms in v/a being negligible. This leads the expression

$$R = A\rho d\nu v$$

viscous drag.

For velocities higher than the critical velocity we have a change physical conditions, the air behind the projectile breaks up into

eddies and the linear law of viscous drag no longer holds.* In su circumstances the resistance is found experimentally to be appro: mately proportional to the square of the velocity.

For incompressible fluids an expression of the form

$$A\nu/vd + B$$

for the drag coefficient will usually fit experimental data for bod completely immersed, A and B being constants depending on t shape of the body When v is very small the first term is large co pared with the second, and the linear law for viscous drag reappea when, on the other hand, v is large the first term becomes sm compared with the second, and we have an approximate quadra law.

An expression of the same form will also hold for the resistar of air to a projectile, provided the velocity is not sufficiently high cause compression of appreciable amplitude. No upper limit velocity can be fixed for this law, since the amplitude of the compr sion will depend on the shape of the head of the projectile as well on the velocity; thus, for Krupp normal shells, which have a mo or-less pointed head, the drag coefficient changes extremely slov with v even at a velocity of 215 m per second, showing that the a proximate quadratic law holds for these projectiles at this veloci whereas with cylindrical projectiles (flat heads) the drag coeffici changes rapidly at this velocity (see fig. 5)

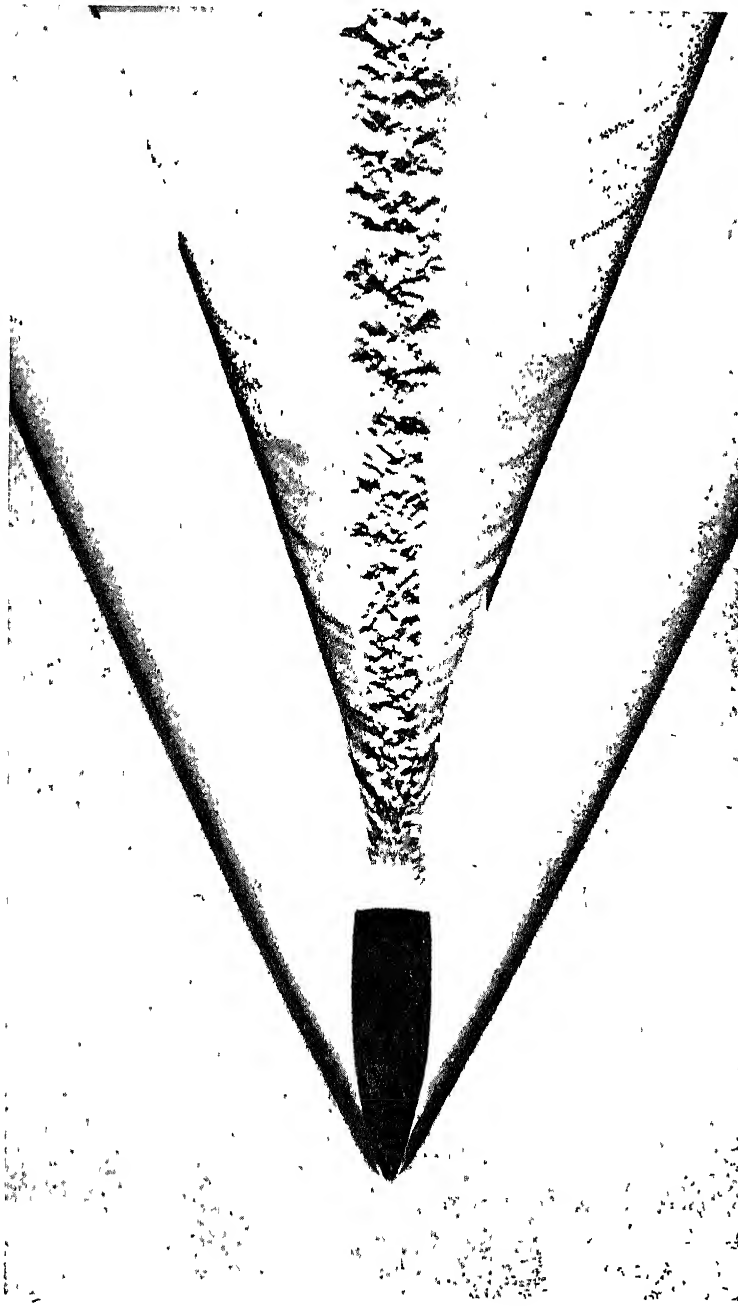
There is little experimental evidence of the behaviour of t drag with variations of d for projectiles at these velocities T results of wind-channel experiments confirm the form given abc for the drag coefficient for velocities up to 30 m. per second, a it has generally been found that the drag coefficient is greater projectiles of small calibre than for those of large calibre of the sa shape.

The Drag at High Velocities

For velocities greater than the velocity of sound the physi conditions are again changed.

At the nose of the projectile the air undergoes condensati The air being an elastic fluid, a condensation formed at any pc in it is transmitted in all directions with a velocity which is, in gene the *velocity of sound.* If, then, the projectile is travelling witl velocity less than that of sound, the condensation of the air at

* See Chapter III, p. 103.



C Cranz (Berlin 1917)

FIG 3 – PHOTOGRAPH OF AN 8 MM BULLET MOVING WITH A VELOCITY OF ABOUT 880 M/S

Facing page 356

ose will be transmitted, as soon as it is formed, away from the nose ı all directions If, on the other hand, the projectile is moving faster ıan sound is propagated, the condensation of the air at the nose cannot e transmitted away from the nose in all directions; it can be transmitted away laterally, but not forwards. The result is that the nose always in contact with a cushion of compressed air. Greatly ıcreased pressure is thus experienced by the projectile when travelling ith a velocity greater than that of sound.

Photographs of bullets moving with such velocities reveal the xistence of two wave fronts, somewhat conical in shape, one at the ead and the other at the base. In fig. 3 a photograph taken by Cranz his ballistic laboratory is reproduced.

The wave front at the head can be accounted for by Huyghens' inciple; it is in fact the envelope of spherical waves which originate the head of the projectile at successive instants of time. If the nplitude of the condensation, when first formed, were small the ave front would be a cone of semi-angle Ω, such that

$$\sin\Omega = a/v. \quad . \qquad .(2)$$

'hen v is less than a the spherical waves have no envelope, and, course, no wave front is formed.

In the actual state of affairs the amplitude of the condensation the nose is not small, but finite. The velocity at which it is opagated is therefore greater than the normal velocity of sound. t points on the wave front near the nose we should therefore expect e angle Ω to be greater than at more distant points where the nplitude has become considerably reduced The form of the tual wave front at the nose is therefore a blunted cone, and the itter the head of the projectile the more is the wave front unted.

The formation of the waves behind the projectile cannot be counted for in such a satisfactory manner. Lord Rayleigh * has own that the only kind of wave of finite amplitude which can be aintained is one of condensation; his argument refers to motion in ıe dimension only, but we see no reason for modifying the result ıen applied to motion in three dimensions. We therefore conclude at the wave at the base of the projectile is, like the head wave, one condensation. An examination of the photographs of bullets in ght verifies this conclusion.

* "Aerial Plane Waves of Finite Amplitude", *Scientific Papers*, Vol. V.

The source of disturbance causing this wave might be identifi with the relatively high state of condensation of the air flowing ı the rarefied region at the base

The angle Ω of the straight part of the wave appears genera to be less than that of the head wave; the difference angle is probably the geometrical consequence of placing t source of light close to the bullet; we have, in fact, a p spective view of the waves. When the source of light very close to the bullet the consequent difference in angle m be considerable.

The tendency of the angle to diminish towards the apex of t wave is probably due to two effects. In the first place there m be some variation in temperature of the air in the immediate neig bourhood. Close to the base the air may be cooler than at poi more distant; the wave may therefore be propagated with less veloc in the vicinity of the base than at more distant points. In the seco place it seems certain that the air behind the projectile will hav velocity gradient from the axis outwards Near the axis the air w be moving faster than at more distant points

Of these two effects the first will tend to diminish a, wh the second will cause an increase in v in equation (2); t values of Ω will therefore be less near the apex of the wave

The change of sign of Ω immediately behind the base is probal due to change in direction of the air's motion in the immediate neig bourhood Lord Rayleigh has proved,* further, that such waves condensation cannot be maintained in the absence of dissipat forces. It is therefore evident that some term involving the viscosi such as vd/ν, must be included in the drag coefficient.

The Scale Effect

There is some experimental evidence of the dependence of the d coefficient on d, and hence on some such term as vd/ν. For examp Cranz† quotes the following figures for the resistance per square cer metre of cross section deduced from Krupp's 1912 experiments:

(*a*) *For cylindrical shell* (flat heads).

Calibre (cm.)	for v = 400	500	600	700	800 m./sec.
6·5	1·40	2·58	3·80	5·15	6·60 Kgm./c
10·0	1·29	2·20	3·30	4·70	6·30 ,,

* Loc. cit. † *Lehrbuch der Ballistik*, Vol. I (Berlin, 1925).

(*b*) *For ogival* * *shell*, 3 *calibres radius*.

Calibre (cm.)	for $v =$ 550	650	750	850 m./sec.
6	1·00	1·30	1·58	1·94 Kgm./cm².
10	0·98	1·25	1·52	1·85 ,,
28	0·62	0·81	1·01	1·25 ,,
30	—	—	0·90	1·06 ,,

In the absence of a term involving d, such as vd/ν, from the drag coefficient the numbers in each vertical column would be equal. The discrepancies are, no doubt, partly due to differences in the yaw of the projectiles, but they cannot be wholly accounted for in this way.

These results indicate that the drag per unit cross section (i.e. $4R/\pi d^2$) for projectiles of small calibre is larger than that for those of large calibre. Didion noticed this so-called scale effect as early as 1856† and deduced a relation between R/d^2 and d, but later he abandoned it as it would not hold for all velocities encountered in gunnery. This effect has not been confirmed in recent experiments and further evidence is needed before definite conclusions can be drawn.

Dependence of the Drag on Density

The assumption that R varies as ρ, other factors being constant, has considerable theoretical support, but up to the present the range of variation of ρ in experiments has been extremely small; we cannot therefore claim practical verification for this assumption. When more work has been done with the high-velocity air stream more light may be thrown on this question, as considerable variations of air density are easily obtained in this method.

The Function $f(v/a,\ vd/\nu)$

At the present time no satisfactory mathematical expression for the drag coefficient has been derived from theoretical considerations. We are therefore forced to accept values of the function derived from experiment alone. In ballistic calculations it has generally been assumed that the term vd/ν could be neglected, that is to say, that the drag coefficient is independent of the calibre; the function $f(v/a,\ vd/\nu)$ has, in consequence, been determined as a function of v/a only.

* See p. 360. † *Lois de la résistance de l'air* (Paris, 1857).

Shape of Projectile

In our discussion we have so far considered the air resistance to projectiles of the *same* shape. Our next step is to consider the changes that occur in the resistance when the shape is altered

All modern projectiles have a cylindrical body and a more-or-less pointed head. The head is usually ogival, that is to say, it is generated by the rotation of an arc of a circle about the axis of the projectile. The shape of the head is identified by the length of the radius (in calibres) of this arc. Thus in fig. 4 a head of 3-calibres radius is depicted

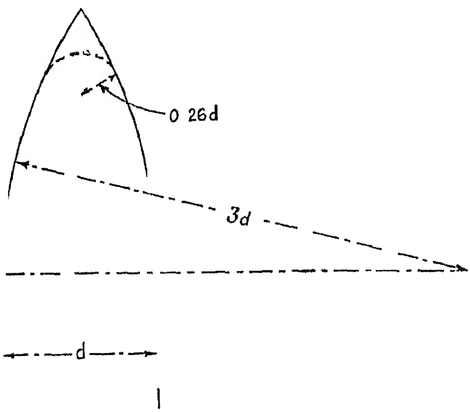


Fig 4 —Shape of Head

When the point is rounded the radius of the rounding is also stated in calibres The dotted head in fig 4 would be described as a 3-calibres radius head with a 0 26-calibre rounded point

In fig 5 the drag co-efficients of a 15-c m. projectile with flat-head and pointed heads of various lengths are plotted against the ratio, velocity of shell to velocity of sound. The curves are deduced from British and foreign experimental data

The values of the drag co-efficient given in these curves may be used with any self-consistent system of units, for example, if the fundamental units used are the metre, kilogramme and second, these values of the drag coefficient when used in equation (1) will give the drag in metre-kilogramme-second units of force (1 unit = 100,000 dynes) Again, if the ft.-lb. and second be used, these values of the drag coefficient used in equation (1) will give the drag in poundals. This property arises, of course, from the fact that $f(v/a)$ has no physical dimensions.

The shape of the head, provided it is more-or-less pointed, does not appear greatly to influence the resistance at lower velocities. At velocities greater than about 350 m. per second, however, the effect of the length of the head is appreciable. At velocities greater than about 750 m. per second it appears that the shape of the point

is more important than that of the rest of the head Thus the resistance is less, at these velocities, for a sharp-pointed 3-calibres radius head than for a 5·5 calibres radius head with a blunted point. For the same shape of point the resistance is less, at velocities greater than about 350 m per second, for a long head (e.g. 5·5 calibres radius) than for a short one (e g. 2-calibres radius).

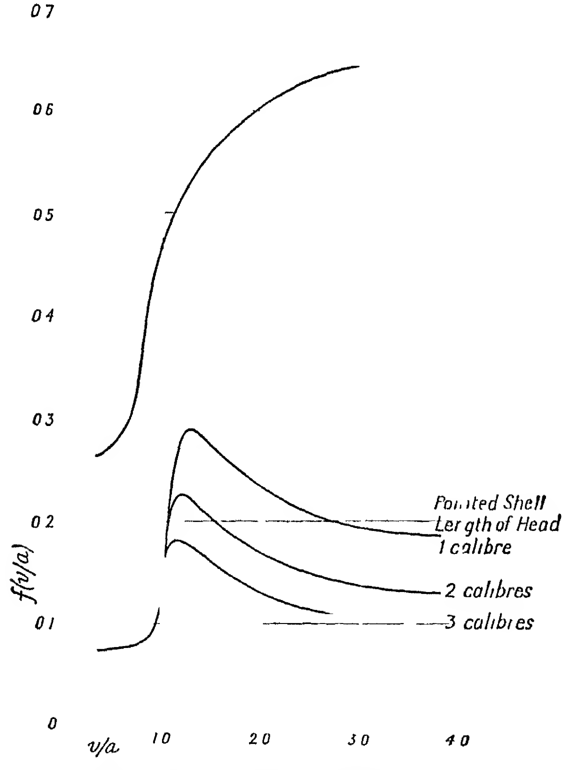


Fig 5 —The Drag Coefficients for 15-cm Projectile with Various Shapes of Head, plotted as a Function of Velocity

Experiments and trials lead to the conclusion that at high velocities long head with a sharp point encounters considerably less resistance than a short head. For example an 8-calibres radius head experiences only about half the resistance of one of 2-calibres radius. Little advantage appears to be gained, however, by lengthening the head beyond 8-calibres ogival radius; thus 10- and 12-calibres radius heads are only slightly more effective than those of 8-calibres radius reducing the air resistance.

The Base

The importance of the shape of the rear part of a body moving in a resisting medium has been realized for many years; the torpedo, the racing automobile, and the fusilage of an aeroplane are examples of " stream-lining " familiar to all. The suggestion has frequently been made that artillery projectiles should have a tapered (so-called " stream-line ") base with a view to reducing the air resistance.

Experiments with rifle bullets have shown, however, that the stability is so seriously affected that any possible advantage gained by a pointed base is entirely eclipsed by the effects of a rapidly developing yaw.

In recent times a compromise has been effected in a shape known as the " boat-tail ", which is illustrated in fig. 6. The base is tapered for a short distance and is then cut off square. The stability of the projectile is not appreciably affected by this modification of the base. Bullets of this shape were tried in France as far back as 1898, but they were found to have no great advantage over the flat base. It has, however, been shown recently that, although such a base has no particular advantage at high velocities, it has appreciable superiority over the flat base at velocities below about 450 m per second. In the trials mentioned above, the French experimented at high velocity over short ranges and so failed to discover the merits of this shape.

Fig 6.—Boat-tail Projectile

Some extremely interesting and suggestive results are recorded by G. P. Wilhelm * of comparative experiments with bullets having the boat-tail and the flat base. The following table gives a summary of these results:

MUZZLE VELOCITY, 1500 FEET PER SECOND

Angle of Departure.	Range (Yards), Flat Base	Range (Yards), Boat-tail
0° 20′	200	220
0° 40′	360	410
1° 0′	500	580
1° 20′	630	720
1° 40′	740	840

* In " Long Range Small Arms Firing ", Part VII, *Army Ordnance*, Washington, March–April, 1922.

MUZZLE VELOCITY, 2600 FEET PER SECOND

Angle of Departure	Range (Yards), Flat Base	Range (Yards), Boat-tail.
0° 20′	570	600
0° 40′	930	990
1° 0′	1050	1200
5° 0′	2250	2700
10° 0′	2900	3600
15° 0′	3250	4200
20° 0′	3500	4650

It is clearly seen that for velocities lower than 1500 ft. per second the boat-tail bullet considerably out-ranges the flat-base bullet. On the high-angle trajectories with a muzzle velocity of 2600 ft per second the velocity of the bullet is, during the greater part of its flight, less than 1500 ft. per second, in fact it is generally only a few hundred feet per second; on the low-angle trajectories, on the other hand, the velocity is less than 1500 ft. per second only in the final stages of the flight. The appreciable gain in range by the boat-tail bullet fired with high muzzle velocity at high angles, and the small or negligible gain at low angles, thus confirms the hypothesis that the shape of the base is of greater importance at low than at high velocities

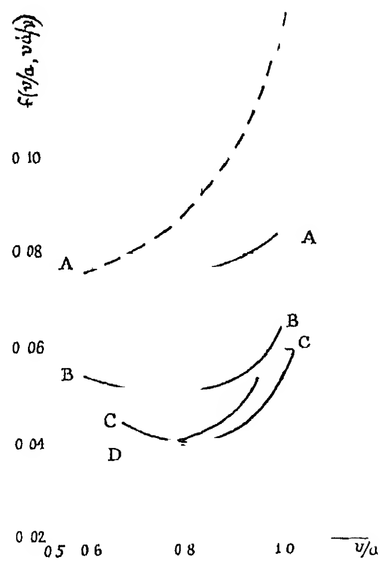


Fig 7—The Drag Coefficient deduced from High-speed Air-stream Experiments

Curve A—For 5-cal -rad head and flat base *Curve B*—Same head as A, boat-tail base, taper, 5° *Curve C*—Same head as A, boat-tail base, taper, 7° *Curve D*—Same head as A, boat-tail base, taper, 9° *Dotted Curve*—The drag coefficient given in fig 5 (flat base).

The results of experiments with the high-speed air stream are interesting in this connection. In fig. 7 some of the results of experiments conducted by the Ordnance Department of America are reproduced.*

* From " Experimental Determination of Forces on a Projectile ", by G. F Hull, *Army Ordnance*, Washington, May–June, 1921.

These curves tend to show that for velocities below 350 m. per second the drag is greatly influenced by the shape of the base; on the other hand, as we have already seen, provided it is more or less pointed, the actual shape of the head has very little influence on the drag at these lower velocities

From these results we may fairly conclude that the greater part of the air resistance to pointed projectiles at these velocities is due to the drag (suction) at the base, and that this drag is appreciably reduced by boat-tailing

The divergence of curve A in fig. 7 from the dotted curve is not clearly understood It is possible that the assumptions made in deducing the velocity of the air stream are not altogether sound and lead to values which are too high, it is also possible that the rod supporting the model (p. 354) may materially affect the air flow at the base and so modify the drag

We have seen that at high velocities the drag is greatly affected by the shape of the head, whereas no appreciable effect is produced by modifying the base. The probable explanation of this is not far to seek. At velocities greater than 750 m per second (the so-called "cavitation" velocity of air) the vacuum at the base must be of high order, and, as the velocity of the projectile increases, it must tend asymptotically to a perfect vacuum. We should therefore expect that the component of the air resistance due to the base is tolerably constant at these high velocities, whereas the total resistance is rapidly increasing with velocity The component due to the base, with increasing velocity, soon becomes a small part of the total resistance, and therefore any possible modification of it, due to shape of base, can have little influence on the whole

Our observations on the effect of shape of pointed projectiles may now be conveniently summarized At velocities less than about 350 m. per second the drag at the base contributes the greater part of the air resistance, so that the shape of the base is of greater importance. At velocities between about 350 m. and 750 m. per second we have an intermediate stage in which the shape of the head gradually gains ascendancy. At velocities greater than about 750 m. per second the greater part of the resistance is due to the head, and the shape of the latter is of greater importance than the shape of the base.

Before leaving the subject of shape we must refer to some extremely interesting experiments designed to determine the pressure distribution on the head of a projectile moving with high velocity.

The Pressure Distribution on the Head of a Projectile

The pressure at any point of a body moving through a fluid consists of two components—the *static pressure*, which is the pressure of the fluid when the body is at rest, and the *dynamic pressure*, which is due to the motion. The sum of these two at any point is the total pressure at that point and is essentially positive; the dynamic pressure may be either positive or negative.

A series of experiments was carried out by Bairstow, Fowler and Hartree to determine the dynamic pressure at various points on the head of a shell moving with high velocity.* The fundamental idea of the experiments is the use of a service time-fuze† as a manometer to determine the pressure under which the powder is burning

Projectiles were fitted with hollow caps which entirely enclosed the fuze, each cap had a number of holes drilled in it at the same distance from the nose, the pressure on the fuze was thus practically equal to that at the holes.

The projectiles were fired along the same trajectory at various fuze settings and times to burst were observed, a relation between fuze setting and time was thus obtained, whence was deduced a relation between rate of burning and time

Since rate of burning is a function of the pressure on the fuze, by comparison with laboratory experiments it is possible to convert this relation to one of pressure and time Knowing the velocity of the projectile at various times of flight on the trajectory, it is thus possible to deduce the pressure in terms of the velocity

By repeating the experiment with other caps of the same size and shape with holes at other distances from the nose, the pressure distribution over the head is obtained at a number of velocities.

The results of the experiments are reproduced in fig 8 The ordinates are values of $p/\rho v^2$, as this quantity has no physical dimensions the values given may be used with any self-consistent system of units. The abscissæ are distances from the nose of the projectile Observations were made at four positions on the head, indicated by

* For a full account of these experiments see " The pressure distribution on the head of a shell moving at high velocities ", *Proc Roy Soc.* A, 97, 1920

† " A service time-fuze contains a train of gunpowder, which is ignited by a detonator pellet on the shock of discharge from the gun The ' setting ' of the fuze can be varied so that a length of powder train depending on the setting is burnt before the magazine of the fuze is ignited and the shell exploded The setting is specified by a number which defines the length of composition burnt on an arbitrary scale The time of burning is taken as equal to the time interval between the firing of the gun and the bursting of the shell."—Loc. cit.

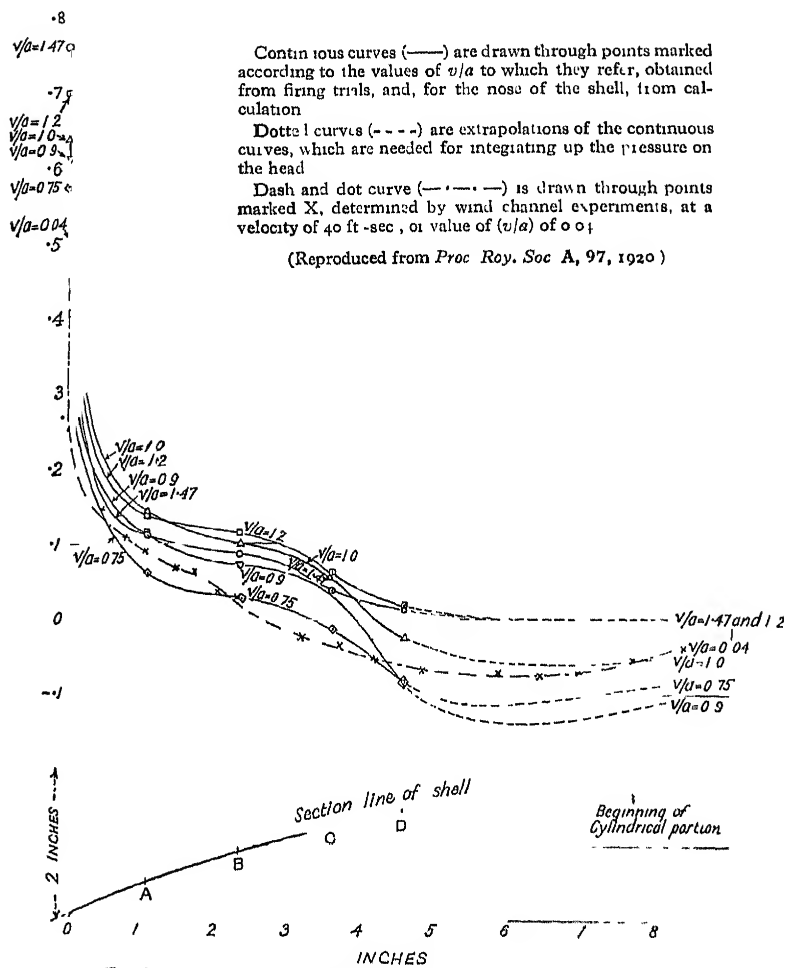


Fig. 8.—Distribution of Pressure on the 6-cal.-rad Head of a 3·3-in Shell

the points A, B, C, and D. The value of $p/\rho v^2$ at the nose was calculated from Rayleigh's formula * in each case.

$$* \frac{p}{\rho v^2} = \frac{a^2}{\gamma v^2}\left[\left\{1 + \tfrac{1}{2}(\gamma - 1)\frac{v^2}{a^2}\right\}^{\frac{\gamma}{\gamma - 1}} - 1\right] \text{ when } \frac{v}{a} \leq 1;$$

$$\frac{p}{\rho v^2} = \frac{1}{\gamma}\left[\frac{\{\tfrac{1}{2}(\gamma + 1)\}^{\gamma + 1}}{\gamma - \tfrac{1}{2}(\gamma - 1)a^2/v^2}\right]^{\frac{1}{\gamma - 1}} - \frac{a^2}{\gamma v^2} \text{ when } \frac{v}{a} \geq 1;$$

deduced from the formulæ given in his *Scientific Papers*, Vol. V, p. 610.

These curves reveal most emphatically the necessity of a sharp point at the nose of the projectile. Compared with the pressure encountered at the nose the pressures at other points of the head are quite small.

The authors integrated numerically the observed pressures over the head in each case, and derived values of the drag coefficient for

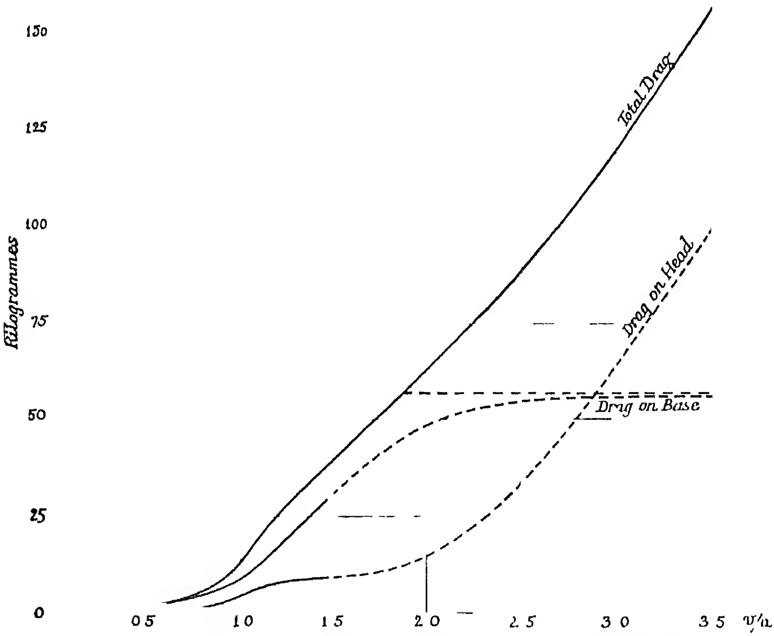


Fig 9 —Drag in Kilogrammes Weight on a 3 3-in 6 C R H Projectile

the dynamic resistance on the head. From these results we have computed the actual dynamic resistance on the head; it is plotted against velocity in fig 9.

The total drag on projectiles of this shape is also shown (approximately) in the figure. By subtracting the head resistance from the total resistance we have derived an approximate curve of the drag at the base The horizontal dotted line indicates the drag due to a complete vacuum at the base in this case, and the dotted extensions of the curves represent a tentative extrapolation of the results of the experiments.

The Effect of Yaw on the Drag

Before approaching the complicated reaction of the air to a yawing projectile it will be convenient to consider, briefly, the effect of yaw on the drag; we now define the latter as the force exerted on the projectile in the opposite direction to the relative motion of the air and the centre of gravity of the projectile.

If δ be the angle of yaw the drag coefficient will now take the form

$$f(v/a,\ vd/\nu,\ \delta),$$

and, in this notation, the drag previously considered takes the form *

$$f(v/a,\ vd/\nu,\ 0)$$

The manner in which the drag coefficient varies with yaw at very low velocities has been determined experimentally in wind channels The results of such experiments on a 3-in projectile with a 2-calibres-radius head and 0 15-calibre rounded point are given in fig 10;† the ordinates are values of the ratio

$$f(v/a,\ vd/\nu,\ \delta)\,/\,f(v/a,\ vd/\nu,\ 0)$$

The velocity at which the experiments were conducted was 40 ft per second ($v/a = 0{\cdot}04$).

We have seen that the drag on a body moving in air is approximately proportional to the square of the velocity,‡ provided that the shape is such that the condensation in front is of small amplitude, for example, this quadratic law holds for pointed projectiles for values of v/a not greater than 0·65 when the yaw is zero

We might reasonably expect the quadratic law to hold for yawing pointed projectiles within the same limit of velocity, provided that the yaw is such that the air is encountered point first, for the condensation would be of the same order as when the yaw is zero Within this limit for yaw, with values of v/a less than 0·65, we should therefore expect that the ratio $f(v/a, vd/\nu, \delta)/f(v/a, vd/\nu, 0)$ is independent of v

* Approximately. Except in the case of results quoted from air-stream experiments we cannot be certain that the values of the drag coefficient hitherto used are for zero yaw All we can affirm is that they are the values for very small or zero yaw.

† This curve is derived from one given in "The Aerodynamics of a Spinning Shell", by Fowler, Gallop, Lock, and Richmond, *Phil. Trans.* A, 591, 1920.

‡ Except, of course, for such low velocities that the drag is due to viscosity alone.

and therefore a function of δ only. The limit of δ for the projectile under consideration appears to be about 45°.

Experiments with the high-velocity air stream tend to verify the independence of velocity of this ratio within the limits mentioned,

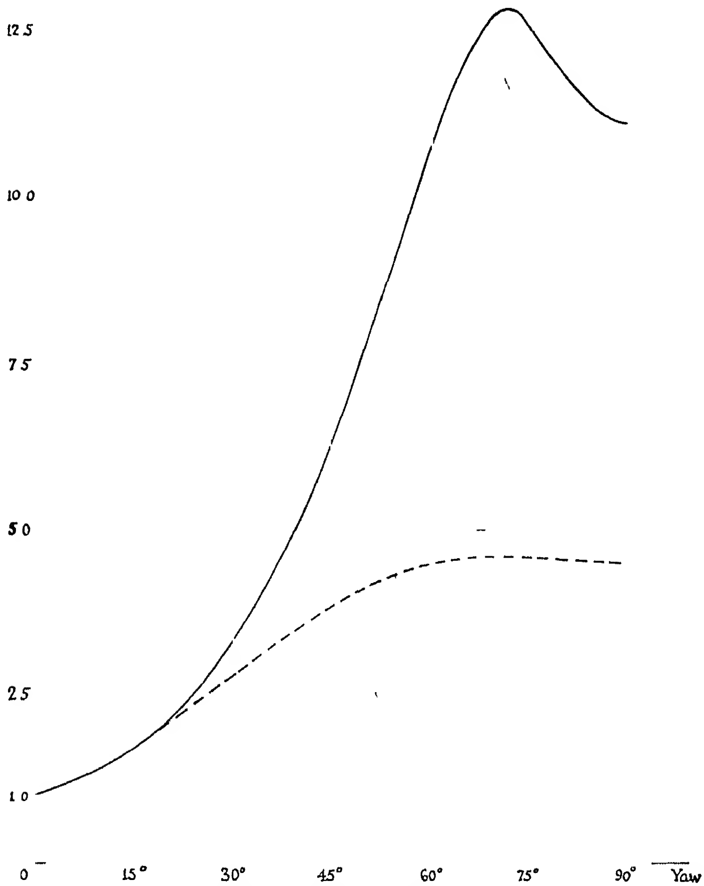


Fig 10 —The Ratio $f(v/a, vd/\nu, \delta)/f(v/a, vd/\nu, 0)$ plotted against the yaw δ for $v/a = 0\ 04$

but at present the number of results available is insufficient to justify our drawing definite conclusions.

The dotted curve in fig. 10 gives, approximately, the ratio between the total plane areas encountered by the projectile with yaws δ and zero. The former is, of course, equal to the area of the shadow cast

by the projectile, in a parallel beam of light inclined at angle δ with the axis, upon a plane normal to the beam; the latter is simply the cross-sectional area of the projectile. The two curves in the figure are coincident for small angles of yaw, but rapidly diverge as the yaw increases; both curves appear to have a maximum at about the same value of the yaw.

We have at present no knowledge of the effect of yaw on the drag at higher velocities; when more work has been done with the high-velocity air stream it is hoped that our knowledge in this direction will have been considerably extended

The Drag Coefficient; Concluding Remarks

In our consideration of the drag coefficient we have limited the number of arguments of the function to three only, namely v/a, vd/ν, and δ. There appear to be two other possible arguments, namely γ, the ratio of the specific heats, and σ/d, where σ is the effective diameter of the molecules of the air.

Variations of γ are so small in practice that no evidence of its effect is available. Expressions deduced from thermodynamic theory, by various authors, for the drag in one-dimensional motion may give an indication of the manner in which it occurs *

There is at present no evidence of the necessity of the argument σ/d. If further experimental results show that the argument vd/ν does not adequately account for the " scale " effect (e g. with varying ν), it will of course be necessary to include some other argument involving d, such as σ/d

Finally, in the case of a projectile moving in air, as distinct from one which is stationary in an air stream of constant velocity, there is the question of retardation. It is just possible that some such argument as rd/v^2, r being the retardation of the projectile, may be required to co-ordinate the results of air-stream experiments with those of experiments on a projectile moving in air, but it is difficult to see how the retardation can ever have an appreciable effect on the drag, except, perhaps, when the velocity is in the neighbourhood of the velocity of sound in air.

* See for example the footnote on p. 346; also Vieille, *Comptes Rendus*, 130 (1900), and Okinghaus, *Monatsh. für Mathem. u. Phys.*, **15** (1904).

REACTION TO A YAWING, SPINNING PROJECTILE

The most complete specification in existence of the system of ɔrces acting on a projectile is that given by Fowler, Gallop, Lock, ıd Richmond in "The Aerodynamics of a Spinning Shell", *Phil. 'rans* A, **591** (1920). In this paper the authors describe experiments ɔnducted to determine numerically the principal reactions, other ıan the drag, to which a spinning shell is subjected

The experiments are confined to the study of the angular oscil- ıtions of the axis of the shell relative to the direction of motion of ıe centre of gravity. The projectile is fired horizontally through a ɜries of cardboard targets fixed, vertically, along the range at 30 ft. ɑd 60 ft. intervals Initial disturbances at the muzzle give rise to scillations of the projectile of sufficient amplitude for measurement; ıe details of the oscillations are obtained by measuring the shape of ıe holes made in the cards If, on passing through a card, the shell ; yawing, the resulting hole will be elongated, the length of the ɔnger axis of the hole determines the yaw; the orientation of this xis determines the azimuth of the plane containing the axis of the hell and the direction of motion; these two angles determine the irection of the shell's axis completely

The range containing the cards is so short, and the velocity so ıgh, that the effect of gravity is negligible If, then, we ignore amping forces, the angular motions of the axis of a top and the xis of a shell are identical, provided that (1) the top and shell have the ame axial spin and axial moment of inertia; (2) the transverse moment f inertia of the top about its point of support is equal to the trans- erse moment of inertia of the shell about its centre of gravity; (3) he moment of gravity about the point of the top is equal to the ɑoment of the force system on the shell about its centre of gravity. 'he formal solutions of the two problems are then identical.

From the periods of the oscillations of the axis of the top we can leduce the moment of the disturbing couple, and *vice versa*; similarly he moment of the force system on the shell can be deduced. The lamping forces can then be determined from the nature of the decay ıf the oscillations.

The reactions are described by the authors as follows:

The Principal Reactions

When the shell, regarded for the moment as without axial spin, has a yaw δ, and the axis of the shell OA and the direction of motion OP remain in the same relative positions, the force system can by symmetry be represented, as shown in fig. 11, by the following components, specified according to aerodynamical usage:

(1) *The drag* R acting through the centre of gravity O, in the direction PO.

(2) A component L, at right angles to R, called the *cross-wind force*, which acts through O in the plane of yaw POA, and is positive when it tends to move O in the direction from P to A

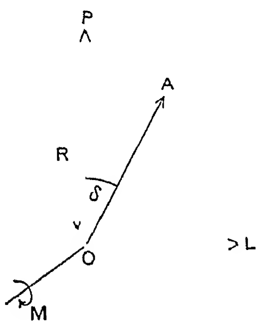


Fig 11.

(3) A moment M about O, which acts in the plane of yaw, and is positive when it tends to increase the yaw.

The following forms are assumed for L and M:

$$L = \rho v^2 d^2 \sin\delta \, f_L(v/a, \delta).$$

$$M = \rho v^2 d^3 \sin\delta \, f_M(v/a, \delta).$$

These equations are of the most natural forms to make f_L and f_M of no physical dimensions. The form chosen is suggested by the aerodynamical treatment of the force system on an aeroplane Since L and M, by symmetry, vanish with δ, the factor $\sin\delta$ is explicitly included in these expressions in order that f_L and f_M may have non-zero limits as $\delta \to 0$.

The Damping Reactions

The yawing moment due to yawing—In practice the direction of the axis of the shell, relative to the direction of motion, changes fairly rapidly. By analogy with the treatment of the motion of an aeroplane, we assume, tentatively, that the components of the force system R, L, and M are unaltered by the angular velocity of the axis, but that the effect of the angular motion of the latter can be represented by the insertion of an additional component, namely, a couple H, called *the yawing moment due to yawing*, which satisfies the equation

$$H = \rho v w d^4 f_H(v/a, \ldots),$$

here w is the resultant angular velocity of the axis of the shell. 'he form is chosen to make f_H of no physical dimensions and is the nly one suitable for the purpose.

The couple is assumed to act in such a way as directly to diminish) (see fig. 12). It is suggested by, and is analogous to, the more nportant of the " rotary derivatives " in the theory of the motion f an aeroplane.

The coefficient f_H may be expected to vary considerably with $/a$, and it may depend appreciably on other arguments such as $\flat d/v$ and δ

The effect of the axial spin.—The spin N gives rise to certain dditional components of the complete force system.

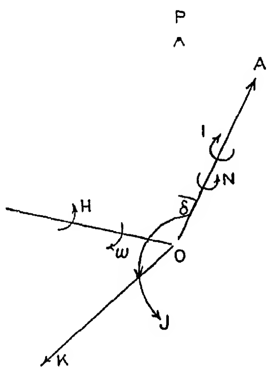


Fig 12

There will be a couple I which tends to destroy N, and, when the shell is yawing, a sideways force, which need not act through the centre of gravity, analogous to that producing swerve on a golf or tennis ball. This force must, by symmetry, vanish with the yaw; it is assumed to act normal to the plane of yaw (any component it may have in the plane of yaw is inevitably included in either R or L). The complete effects of the spin N can therefore be represented by the addition to the force system of the couples I and J and the force K, acting as shown in fig. 12.

To procure the correct dimensions we may assume that these reactions have the forms

$$I = \rho v N d^4 f_I.$$
$$J = \rho v N d^4 \sin\delta\, f_J.$$
$$K = \rho v N d^3 \sin\delta\, f_K.$$

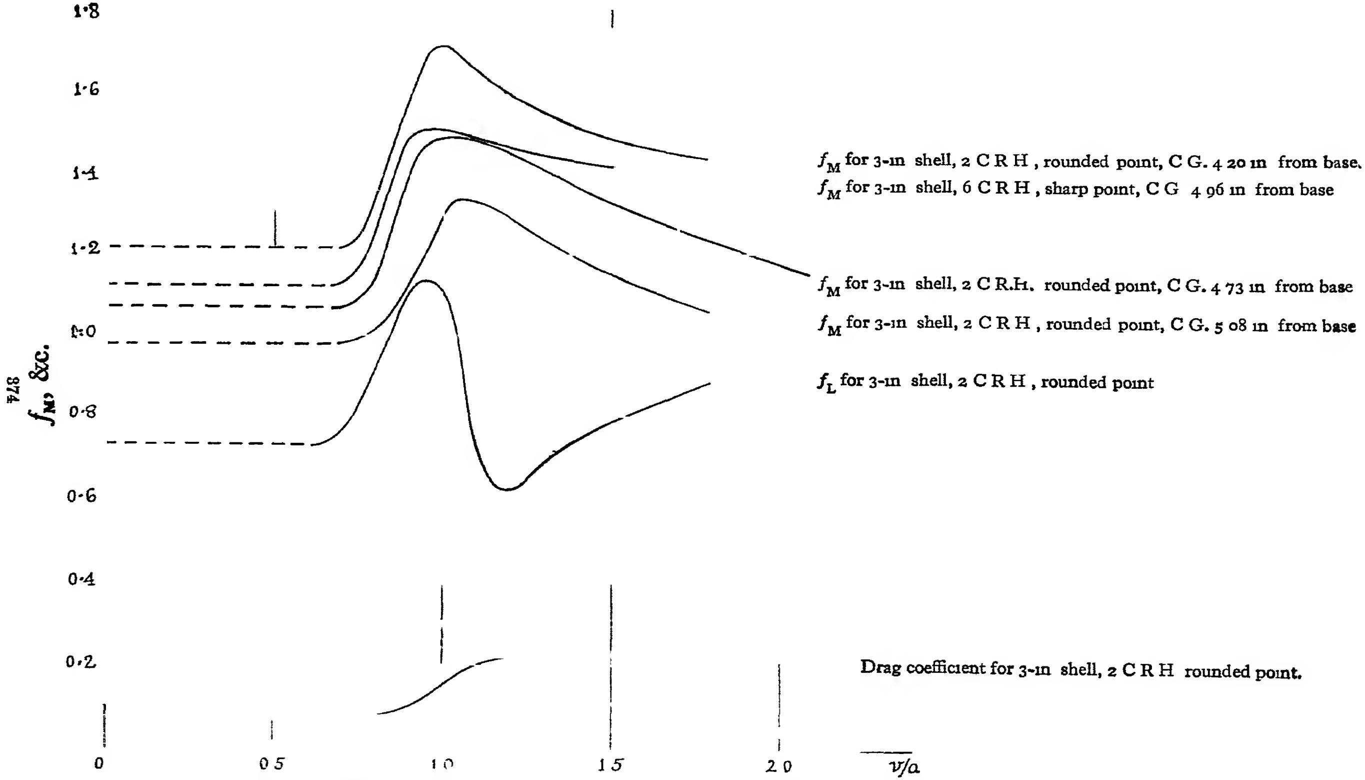


Fig 13 —$f_M(v/a, 0), f_L(v/a, 0)$ and the Drag Coefficient plotted against v/a (From *Phil Trans* A, 591, 1920)

The coefficients f_I, f_J, f_K may depend effectively on a number of ıables which we can make no attempt to specify in the present te of our knowledge.

It will be seen that this specification is equivalent to a complete ;tem of three forces and three couples referred to three axes at ,ht angles. Owing to the complex nature of the reactions the thors considered it essential to construct the specified system

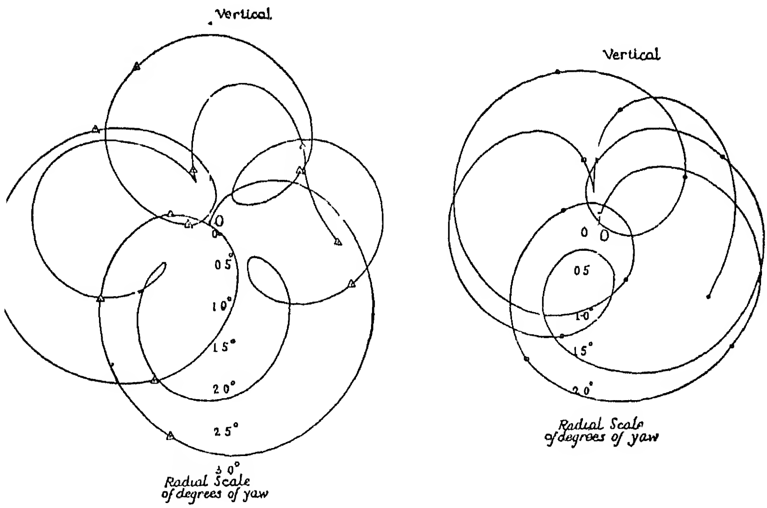


Fig 14 —Examples of Path of Nose of Shell relative to the Centre of Gravity

(From *Phil Trans* A, 591, 1920)

stead of attempting to analyse a complete system of three forces ıd three couples and to assign each component to ıts proper causes.

The experiments were designed to determine L * and M and to ve an indication of the magnitude of the chief damping couple . It was not possible to determine I, J, and K; all three are resumably small compared with H and very small compared with , and M; no certain evidence that they exist was given by the xperiments.

The results of the experiments are exhibited in fig 13, which is :produced from the paper. The units in which the coefficients are xpressed are suitable for use with any self-consistent system. The

* Values of L were deduced from values of M for shell of the same external ıape, with centres of gravity at different positions along the axis of figure.

curve of the drag coefficient is also given; in comparing this with f_L and f_M it must be remembered that the latter should be multiplied by $\sin\delta$. The shape of the curve for f_L is rather unexpected; the curves for f_M appear to exhibit the same tendencies as that for the drag coefficient.

The values of f_H deduced from the experiments were rough; they varied from about 1·4 to 5·0. It was impossible to deduce any details concerning the variations of f_H with velocity, but the right order of magnitude is represented by these limits In comparing f_H with f_M it must be remembered that the former is multiplied by wd, whereas the latter is multiplied by the much larger quantity v.

The general features of the motion of the axis of the shell and of the damping are shown in the examples, reproduced from the paper cited, in fig. 14.

SUBJECT INDEX

(*See also Name Index*)

NAME INDEX